Stat. & prob. Rem. Theorem
Sum & products of quad 27
Discriminant "
Arithmetical & Geo.
Math Induction
Binomial Theorem

62 W.H. Prescott

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COLLEGE ALGEBRA

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PREFACE

This book is designed primarily for use as a text-book in the freshman year of colleges and technical schools. Special attention is directed to the following features:

(1) The method of reviewing the algebra of the secondary schools.

(2) The selection and omission of material.

(3) The explicit statement of assumptions upon which the proofs are based.

(4) The application of algebraic methods to physical problems.

For the majority of college freshmen, a considerable period of time elapses between the completion of the high school algebra and the beginning of college mathematics. The review of the secondary school algebra is written for these students. This part of the book is, however, more than a hasty review. While the student is reviewing a first course, he is at the same time making a distinct advance by seeing the subject-matter from new view points, which his added maturity enables him to appreciate. For example, the functional notation, graphs, and determinants are introduced and used to advantage in the review. The extensions of the number concept receive fuller treatment than is usual in a college algebra. The various classes of numbers from positive integers to complex numbers are treated in the order in which they are demanded by the equation.

The application of algebra in the more advanced courses in mathematics has been an important factor in determining the subject-matter. Not only are some of the topics usually treated in the traditional course in algebra entirely omitted, but in each chapter the material is restricted to the development of those central points which experience has shown to be essential. While a complete discussion of limits and infinite series does not properly belong in a course in algebra, it has been thought best to

include an introduction to these subjects which covers in considerable detail only the theory necessary to a discussion of the comparison and ratio tests. From the experience of the authors, a great deal is gained by thus taking a very elementary first course in limits and series.

While it is out of place in a book of this character to attempt a critical study of fundamentals, great care has been taken to point out just what is proved and what is assumed in so far as a first-year student can be expected to appreciate the necessity of assumptions.

Without trying to teach physics or engineering, many problems are introduced in which the principles of algebra are applied to physical problems, but no technical knowledge is assumed on the part of the student. Rules for the mechanical guidance of students in solving problems have been used sparingly.

The authors take great pleasure in acknowledging their indebtedness to their colleagues in the mathematical and engineering departments of the University of Illinois. We are indebted to Professors Haskins and Young for suggestions during the preparation of the manuscript as well as for a critical reading of the manuscript; to Professors Townsend, Goodenough, Miller, Wilczynski, Dr. Lytle, and to Professor Kuhn of Ohio State University for suggestions upon the manuscript; to Professor Watson for some of the practical problems; and again to Professor Goodenough for assistance in seeing the book through the press.

H. L. RIETZ.
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PREFACE TO REVISED EDITION

Suggested by experience in the class-room, a number of topics have been simplified in treatment. In response to requests of numerous teachers, several hundred new exercises and problems have been introduced. The book has thus been freshened with regard to both text and problems without however changing its general character.

H. L. R.
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A LIST OF SIGNS AND SYMBOLS

$+$, read *plus*. $-$, read *minus*.

$\times$, or $\cdot$, read *times*.

$\div$, read *divided by*.

$=$, read *is equal to*.

$\equiv$, read *is identical with*.

$\neq$, read *is not equal to*.

$\rightarrow$, read *approaches*.

$<$, read *is less than*.

$>$, read *is greater than*.

$\leqq$, read *is less than or equal to*.

$\geqq$, read *is greater than or equal to*.

$a!$ or $\lfloor a$, read *factorial a*.

( ) Parentheses.
[ ] Brackets.
{ } Braces.
—— Vinculum.
| Bar.

Signs of aggregation. These signs are used to collect together symbols which are to be treated in operations as one symbol.

a_r, read *a subscript r*, or *a sub r*.

x', x'' ···, read *x prime*, *x second* ··· respectively.

$\lim x$, read *limit of x*.

$x \rightarrow \infty$, read *x becomes infinite*, or *x increases beyond bound*.

$\log_a n$, read *logarithm of n to the base a*.

$|a|$, read *absolute value of a*.

a^n, read *a to the nth power*, or *a exponent n*.

$\sqrt{a}$, read *square root of a*.

$\sqrt[n]{a}$, read *nth root of a*.

$f(x)$, $\phi(x)$, etc., read "*f*" *function of x*, "ϕ" *function of x*, etc.

${}_nP_r$, read *number of permutations of n things taken r at a time*.

${}_nC_r$, read *number of combinations of n things taken r at a time*.

(x, y), read *point whose coordinates are x and y*.

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

Δ.Κ.Ε. 1919
Ο.Κ.Π. 1919

COLLEGE ALGEBRA

CHAPTER I

INTRODUCTION

1. Numbers. In counting the objects of a group the child makes his first acquaintance with numbers. These are the numbers called positive integers.

He next employs the number we call a rational fraction, probably thinking of it first as an aliquot part, and later as the quotient of two integers. Perhaps in the fall of a thermometer below zero, the student had his first experience in the use of negative numbers, even if he was not taught to use the word negative. He may also come early to the convenient use of the negative number to represent debit when the corresponding positive number means credit.

To express the length of the diagonal of a square of side one unit, or to find a number which multiplied by itself gives some integer, not a perfect square, say 2, he uses a number which is neither an integer nor a rational fraction, and employs a radical sign to represent it by writing $\sqrt{2}$ where $\sqrt{2} \times \sqrt{2} = 2$. Such numbers belong to a class of numbers known as irrational numbers. (See p. 23.)

2. Graphical representation of real numbers. The four classes of numbers mentioned in Art. 1 belong to the so-called "real numbers" used in arithmetic and algebra. They may be represented by the points of a straight line as follows: Let $X'X$ be this line. (Fig. 1.) Choose a point O on this line and call it the zero point or origin. Adopt some unit of measurement OA.

Beginning at O and proceeding in both directions, apply the unit of measure to divide OX and OX' at equal intervals, thus

forming a scale of indefinite length. The positive and negative integers may then be conveniently represented by the points marking the intervals. Similarly, corresponding to any fraction $\frac{a}{b}$ (a and b integers), there can be constructed a point on $X'X$ such that the fraction denotes the distance and direction of the point from O. In fact, we assume that by means of this scale we are able to represent conveniently all real numbers, and we

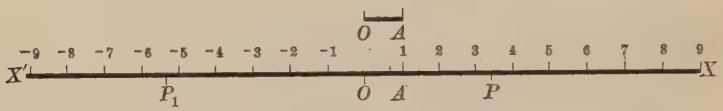

Fig. 1.

can say, to any point P on the line, there corresponds a number * which indicates the distances and direction of P from O, and conversely, we assume that to every real number there corresponds a point of this line.

In addition to the real numbers, we shall find it desirable to deal with so-called "imaginary" and complex numbers. A graphical representation is given for these numbers in Art. 80.

EXERCISES

1. What numbers are represented by the following points?

(a) The point midway between 4 and 5.

(b) Points of trisection of the interval -3 to -4.

(c) Points of quadrisection of the interval 2 to 3.

2. State in words the position of points which represent $\frac{1}{2}$, $\frac{1}{3}$, $-\frac{3}{4}$, -2, π.

3. Suppose the scale of Fig. 1 represents the scale of a Fahrenheit thermometer; estimate the readings when the end of the mercury column stands at P_1.† At P. At point midway between O and P. At point midway between P and P_1.

4. A square piece of paper of side 4 is laid on Fig. 1 so that one corner of the square is at O and the diagonal lies on the line OX. What number is represented by the point at the other corner of the square which lies on OX?

---

* For a more complete discussion, see Fine's *Number System of Algebra*, Second edition, p. 41.

† P_1 is read "P sub one." See list of signs and symbols at the end of the table of contents.

5. A circle of radius 1 rolls to the right along the line in Fig. 1, beginning at O. What number is represented by the point at which the circle touches the line after one complete turn?

3. Greater and less. The terms *greater than* and *less than* which are common to everyday life, when used in the technical sense of algebra, are easily misunderstood. For this reason we point out their geometrical significance. The real number A is said to be greater than the real number B (written $A > B$) if the point representing A falls to the right of the point representing B. The number A is said to be less than the number B (written $A < B$) if the point representing A falls to the left of that representing B.

Exercise. Arrange the following numbers in descending order of magnitude:

$$2, \quad \sqrt{2}, \quad -4, \quad -\sqrt{3}, \quad 3\sqrt{5}, \quad -10, \quad -20, \quad \tfrac{1}{2}.$$

4. Definition and assumptions. Operations with numbers of arithmetic suggest certain definitions and rules for algebra. The student has probably performed algebraic operations according to rules thus suggested by arithmetic without being conscious of the assumptions which underlie these processes. We may now proceed to a formal statement of assumptions made at the outset in this algebra.

Letters are used to represent numbers; a number, which is represented by a certain letter, is called the **value** of the letter. In the following, let a, b, c represent any numbers.

The fundamental* operations of **addition** and **multiplication** of numbers are subject to the following laws I–IX:

I. *The sum† of two numbers is a uniquely determined number.*

That is, given a and b, there is one and only one number x such that $a + b = x$.

II. *Addition is commutative.*

That is, $$a + b = b + a.$$

* The operations are fundamental in that no attempt is made to define them. The "laws" are in the nature of assumptions since no attempt is made to prove them.

† The **sum** is the result of adding; the **product** is the result of multiplying.

III. *Addition is associative.*

That is, $(a + b) + c = a + (b + c)$.

IV. *If equal numbers be added to equal numbers, the sums are equal numbers.*

That is, if $a = b$,
and $c = d$,
then $a + c = b + d$.

V. *The product of any two numbers is a uniquely determined number.*

That is, given a and b, there is one and only one number y such that $ab = y$. In this case, a and b are said to be *factors* of y.

VI. *Multiplication is commutative.*

That is, $ab = ba$.

VII. *Multiplication is associative.*

That is, $(ab)c = a(bc)$.

VIII. *Multiplication is distributive with respect to addition.*

That is, $a(b + c) = ab + ac$.

IX. *If equal numbers be multiplied by equal numbers, the products are equal numbers.*

That is, if $a = b$,
and $c = d$,
then $ac = bd$.

The following laws X and XI lead us to definitions of subtraction and division, and enable us to give meanings to the symbols 0, $-a$, $\frac{a}{b}$, 1, and $\frac{1}{b}$.

X. *Given a and b, there is one and only one number x, such that $x + b = a$.*

Subtraction is the process of finding the number x in $x + b = a$. In other words, to **subtract** b from a is to find a number x, called the **remainder**, such that the sum of x and b is a.

By X, the number x in $x + b = a$ exists when $a = b$. In this case, the number x is called **zero**, and is written 0.

In symbols,

$$0 + b = b. \tag{1}$$

From X and the definition of 0, there exists a number x, such that

$$x + b = 0.$$

In this case, x and b are said to be negatives of each other, and x may be replaced by $(-b)$.

If b is a positive number, x is a **negative** number.

In symbols,

$$(-b) + b = 0, \tag{2}$$

gives a definition of $(-b)$.

XI. *Given a and b ($b \neq 0$*), there is one and only one number x, which satisfies $bx = a$.*

Division is the process of finding the number x in $bx = a$. In other words, to **divide** a by b is to find a number x, called the **quotient** of a by b, such that b multiplied by x gives a.

This quotient is often written $\frac{a}{b}$, and, when thus written, is called a **fraction.**

In dividing a by b, the number a is called the *dividend* and b the *divisor*, just as in arithmetic. Likewise, in the fraction $\frac{a}{b}$, a is called the *numerator* and b the *denominator.*

By XI, the number x in

$$bx = a \quad (b \neq 0)$$

exists when $b = a$, so that $ax = a$. In this case, the number x is called *unity* and is written 1, that is,

$$\frac{a}{a} = 1. \tag{3}$$

Further, by XI, and the definition of 1, there exists a number x, which satisfies $bx = 1$ $(b \neq 0)$. This value of x is called the *reciprocal* of b and is written $\frac{1}{b}$.

---

* The sign $\neq$ stands for "*is not equal to.*"

It should be noted that, by means of XI, a meaning is given to unity and to the reciprocal of any number other than zero in a manner analogous to that by which a meaning is given to zero and to the negative by means of X.

The above propositions I–XI are stated for two and three numbers for the sake of simplicity. It can be proved from the given assumptions that these propositions hold when three or more numbers are concerned in the process of addition or multiplication.

It is not to be inferred that these propositions I–XI are entirely independent of one another, but rather that they constitute convenient assumptions for the purposes of this course in algebra.

Although further assumptions are made later in the course, the principles I–XI enable us to prove many important propositions of algebra, and the wide application of algebra depends upon the fact that many changes in the physical world take place in accordance with these laws.

As shown in the following exercises, the operations of algebra are generalizations of the operations of elementary arithmetic.

EXERCISES

1. Show that 4×2, 9×7, 1×-1, -4×-6, 0×-2, $\frac{1}{2} \times -\frac{3}{2}$, are special cases of $(x+1)(x-1)$ and of $x(x-2)$.

2. Show that $2 \times 3 \times 8$, $5 \times 6 \times 11$, $\frac{1}{2} \times \frac{3}{2} \times \frac{1}{2}^3$, $-3 \times -2 \times 3$, $-1 \times 0 \times 5$ are special cases of $x(x+1)(x+6)$ and of $(x-1)x(x+5)$.

3. Show that $\frac{12}{2} = 6$, $\frac{-4}{-2} = 2$, $\frac{21}{-7} = -3$, $\frac{-3}{-1} = 3$, $\frac{96}{8} = 12$, $\frac{96}{-12} = -8$ are special cases of $\frac{x^2-4}{x-2} = x+2$.

In the course of operations with the numbers of algebra, the important question arises: Can any two given numbers be added, subtracted, multiplied, or divided? Our assumptions state that the number system of algebra is such that this question can be answered in the affirmative **except in the case of division by zero.** *Division by zero is excluded from algebraic operations.*

EXERCISES

1. Can any two given numbers be added, subtracted, multiplied, or divided if the number system consists of positive integers only? Illustrate your answer.

2. Can any two given numbers be added, subtracted, multiplied, or divided if the number system consists of positive integers and quotients of positive integers only? Illustrate your answer.

3. Where is the flaw in the following?

Let	$x = a \quad (x \neq 0).$	(1)
Multiply both sides by x,	$x^2 = ax.$	(2)
Subtract a^2 from both sides,	$x^2 - a^2 = ax - a^2.$	(3)
Factor,	$(x - a)(x + a) = a(x - a).$	(4)
Divide both sides by $x - a$,	$x + a = a.$	(5)
But, by (1),	$x = a.$	(6)
By (5) and (6),	$2a = a.$	(7)
Hence,	$2 = 1.$	

5. Derived properties of the numbers of algebra. From the foregoing definitions and assumptions, the following propositions can be proved. We shall present in detail the proofs of only a few of them.

I. *Adding a negative number* $(-a)$ *is equivalent to subtracting a positive number* a.

That is, $b + (-a) = b - a.$

To prove this, let	$b + (-a) = x.$	(I, Art. 4.)	(1)
	$b + (-a) + a = x + a.$	(IV, Art. 4.)	(2)
But	$(-a) + a = 0.$	(Eq. (2), page 5.)	(3)
From (2) and (3),	$b + 0 = x + a.$		(4)
But	$b + 0 = b.$	(Eq. (1), page 5.)	(5)
From (4) and (5),	$b = x + a.$		(6)
	$b - a = x.$	(Def. of subtraction.)	(7)

That is, $b + (-a) = b - a.$

II. *Subtracting a negative number* $(-a)$ *is equivalent to adding the positive number* a.

That is, $b - (-a) = b + a.$

III. *The product of two numbers is* 0 *when and only when at least one of the numbers is* 0.

COROLLARY. *The quotient* $\frac{0}{a}$ *is equal to* 0 *when* a *is any number different from* 0.

IV. *The product of a number a by a number* $(-b)$ *is* $-ab$.

To prove this, let $a(-b)=x$. (x exists by V, Art. 4.) (1)

Then $a(-b)+ab=x+ab$. (IV, Art. 4.) (2)

$a[(-b)+b]=x+ab$. (VIII, Art. 4.) (3)

$a\cdot 0=x+ab$.* (Eq. (2), page 5.) (4)

$0=x+ab$. (III, Art. 5.) (5)

$-ab=x$. (Definition of negative.) (6)

From (1) and (6), $a(-b)=-ab$.

V. *The product of* $(-a)$ *by* $(-b)$ *is* ab.

To prove this, let

$(-a)(-b)=x$. (1)

$(-a)(-b)+a(-b)=x-ab$. (IV, Art. 4; IV, Art. 5.) (2)

$(-b)[(-a)+a]=x-ab$. (VIII, Art. 4.) (3)

$-b\cdot 0=x-ab$. (Definition of zero.) (4)

$0=x-ab$. (Why?) (5)

$ab=x$. (Why?) (6)

From (1) and (6), $(-a)(-b)=ab$.

The statement that in multiplication *like signs give plus and unlike signs give minus* includes IV and V.

VI. *The quotient of two numbers is positive if the signs of the dividend and divisor are alike; negative if they are unlike.*

VII. *A single parenthesis may be removed when it is preceded by a positive sign without changing the signs of the terms within it.*

VIII. *A single parenthesis may be removed, when preceded by a negative sign if the signs of terms within it are changed.*

That is, $-(a+b-c+d-e)=-a-b+c-d+e$.

IX. *The value of a fraction is not changed by multiplying or dividing both the numerator and denominator by the same number.*

That is, $\frac{a}{b}=\frac{ax}{bx}$

* The · is a sign of multiplication, thus, $5\cdot 6=30$.

X. *Changing the sign of either the numerator or the denominator of a fraction is equivalent to changing the sign of the fraction.*

That is, $$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}.$$

XI. *Adding two fractions having a common denominator gives a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator.*

That is, $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Likewise, $$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

XII. *The sum and the difference of two fractions are expressed by*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \text{ and } \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}, \text{ respectively.}$$

We can reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator, since, by IX,

$$\frac{a}{b} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{bc}{bd}.$$

By XI, we can complete the process.

XIII. *The product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.*

That is, $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

XIV. *To divide one fraction by another, invert the latter and then multiply one by the other.*

That is, $$\frac{a}{b} \div \frac{c}{d} \text{ or } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

Propositions III, XI, XIII, stated for two numbers in each case, are readily extended to three or more numbers.

INDEX LAWS

6. Positive integral exponents. Definition. — The expression a^x is read "a *exponent* x" or the "xth power of a." When x is a positive integer, a^x is a short way of writing $a \cdot a \cdot a \cdots$ to x factors.

Laws: I. $$a^m a^n = a^{m+n}.$$

For, if m and n are positive integers, by the associative law of multiplication,

$$\begin{aligned} a^m a^n &= (a \cdot a \cdot a \cdots m \text{ times})(a \cdot a \cdot a \cdots n \text{ times}) \\ &= a \cdot a \cdot a \cdots m + n \text{ times} \\ &= a^{m+n}. \end{aligned}$$

Illustrations: $5^3 \cdot 5^4 = 5^7$, $3 \cdot 3^2 \cdot 3^3 \cdot 3^5 = 3^{11}$.

II. $$(a^m)^n = a^{mn}.$$

Illustration: $$(3^4)^5 = 3^{20}.$$

III. $$(abc\cdots)^m = a^m b^m c^m \ldots.$$

Illustration: $$(3 \cdot 5 \cdot 7 \cdot 4)^2 = 3^2 \cdot 5^2 \cdot 7^2 \cdot 4^2.$$

IV. $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Illustration: $$\left(\frac{3}{7}\right)^3 = \frac{3^3}{7^3}.$$

V. $$\frac{a^m}{a^n} = a^{m-n}, \quad (m > n).$$

Illustration: $$\frac{5^6}{5^4} = 5^2.$$

VI. $$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad (m < n).$$

Illustration: $$\frac{5^4}{5^6} = \frac{1}{5^2}.$$

EXERCISES

What numbers should be written in the blank parentheses in the following?

1. $3^{\square} \cdot 3^2 \cdot 3^2 = 3^{(\square)}$.
2. $(2^3)^3 = 2^{(\ )}$.
3. $\left(\frac{4^8}{4^5}\right)^2 = 4^{(\ )}$.
4. $\frac{5^4 \cdot 9^3 \cdot 2^5}{5 \cdot 9 \cdot 2^2} = 5^{(\ )} \cdot 9^{(\ )} \cdot 2^{(\ )}$.
5. $\left(\frac{3}{2}\right)^5 \cdot \frac{(2^2)^3}{3} \cdot 2^4 = 2^{(\ )} \cdot 3^{(\ )}$.
6. $(3 \cdot 5 \cdot 9)^2 \cdot (5 \cdot 9)^3 = 3^{(\ )} \cdot 5^{(\ )}$.

Perform indicated operations and simplify when possible.

7. $\frac{a^2 \cdot a^3}{a^4}$.
8. $\frac{x^3 \cdot x^7}{(x^5)^2}$.
9. $\left(\frac{x^2y}{xy^2}\right)^5$.
10. $mn\left(\frac{m^2n^3}{mn}\right)^3$.
11. $\left(\frac{16\,a^2b^4}{(4\,ax)^3}\right)^2$.
12. $(2\,a + b^2)^3$.
13. $a^{2n} \cdot a^n$.
14. $a^{n-1} \cdot a^{n+1} \cdot a^n$.
15. $\frac{a^{2n}}{a^n}$.
16. $a^{2n} \cdot a^2$.
17. $\frac{a^{2n}}{a^2}$.
18. $\left(\frac{ab^2}{a^2b}\right)^n$.
19. $\left(\frac{-3\,x^3y}{4\,xy^2}\right)^n$.
20. $\left(\frac{-3\,x^3y}{4\,xy^2}\right)^{2n}$.
21. $\frac{14\,m^7n^5 + 7\,m^5n^7}{7\,m^6n^6}$.
22. $(x^7y^8 + x^6y^7 + x^5y^6) \div (x^5y^6)$.
23. $(a^n + b^n)(a^n - b^n)$.
24. $(x^{n-1} + y^{n-1})(x^{n-1} - y^{n-1})$.
25. $(p^{n-2} + p^{n-1})(p^n - p^{n-3})$.
26. $\left(\frac{(a^2x^4y^3)^n}{a^3xy}\right)^3$.
27. $\left(\frac{x^5p^7y^{n+1}}{x^2py}\right)^n$.
28. $\left(\frac{x^3x^{2n-1}}{x^{2n}}\right)^3$.
29. $\left(\frac{a^{n+1}}{a^n}\right)^n \div a^n$.
30. $\left(\frac{7\,a^3}{15\,x^3a^{m-1}} \div \frac{a^{n+1}}{a^{m+n}}\right) \frac{x^n}{a^{m-1}}$.
31. $\left(\frac{a^2a^{2n-2}}{a^n}\right)^3$.
32. $\frac{(x^n)^{2n} - (y^{2n})^n}{(x^n)^n - (y^n)^n}$.
33. $\frac{8\,a^{3n} - b^{3n}}{2\,a^n - b^n}$.
34. $\frac{x^{3n} + 1}{x^n + 1}$.
35. $\left(\frac{x^2 - y^2}{a - b}\right)^2 \cdot \frac{(a^3 - b^3)^2}{(x - y)^2}$.
36. $(3\,p^n - 2\,q^n)(9\,p^{2n} + 6\,p^nq^n + 4\,q^{2n})$.

37. Establish index laws II, III, IV, V, VI, stating at each step the principle used.

38. Translate the six index laws from algebraic symbols into English.

7. Meaning of $a^{\frac{1}{q}}$. The proofs of the above six laws of indices assume that the exponents are positive integers. According to the definition of a^x (Art. 6), such an expression as $a^{\frac{1}{3}}$ has no meaning whatever. If we use such expressions, we must first give them a meaning. It is convenient to define them in such a way that the laws for positive integral exponents hold also for fractional exponents.

Assuming Law I, Art. 6, to hold, we shall have

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

Assuming that some number exists whose third power is a, we will denote it by $a^{\frac{1}{3}}$. Another way of writing $a^{\frac{1}{3}}$ is $\sqrt[3]{a}$, which is read "the cube root of a." In general, if q is a positive integer,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots \text{ to } q \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\cdots \text{ to } q \text{ terms}} = a,$$

and $a^{\frac{1}{q}}$ means a number whose qth power is a. Another way of writing $a^{\frac{1}{q}}$ is $\sqrt[q]{a}$, which is read "the qth root of a."

Thus, the fractional exponent $\frac{1}{q}$ serves the same purpose as the radical sign $\sqrt[q]{\ }$.

8. Meaning of $a^{\frac{p}{q}}$. According to Law I, Art. 6, if p and q are positive integers,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots \text{ to } p \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\cdots \text{ to } p \text{ terms}} = a^{\frac{p}{q}},$$

and $a^{\frac{p}{q}}$ means **the pth power of the qth root of a.**

That is, $$a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

9. It will be seen later (Art. 86) that any number a has q distinct qth roots. Thus, the number 4 has the two square roots ± 2. We shall, however, for the present, consider only the arithmetical or positive value of $a^{\frac{1}{q}}$ when a is a positive number.

With this limitation, it turns out that $a^{\frac{1}{q}}(a > 0)$ has one and only one value. For example, $9^{\frac{1}{2}} = 3$, and not ± 3. Likewise, $\sqrt[4]{81} = 3$, and not ± 3. If both the positive and negative roots are meant, we shall write both signs before the radical.

Without this limitation, it will be seen from the following illustration that the pth power of a qth root of a number is not necessarily equal to a given qth root of the pth power:

$(4^{\frac{1}{2}})^4 = 16$, while the square root of 4^4 may be either $+16$ or -16.

10. Meaning of a^0. In order that the first index law may hold for an exponent zero, it is necessary that

$$a^0 \cdot a^m = a^{0+m} = a^m, \tag{1}$$

or,
$$a^0 = 1, \text{ if } a \neq 0.$$

That is, *any number a with the exponent 0 is equal to 1, provided $a \neq 0$.*

11. Meaning of a^m when m is negative. Let $m = -n$, where n is a positive number. By Law I, Art. 6, and Art. 10,

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1. \tag{1}$$

Hence,
$$a^{-n} = \frac{1}{a^n}, \text{ if } a \neq 0.$$

$$a^m = \frac{1}{a^{-m}}. \tag{2}$$

That is, *in a fraction, any factor of the numerator may be transferred to the denominator, or vice versa, if the sign of the exponent of the factor is changed.*

EXAMPLE: $\dfrac{2^2 \cdot 3^{-4}}{7 \cdot 5^{-2}} = \dfrac{7^{-1} \cdot 5^2}{2^{-2} \cdot 3^4} = \dfrac{2^2}{7 \cdot 5^{-2} \cdot 3^4} = 2^2 \cdot 7^{-1} \cdot 3^{-4} \cdot 5^2.$

We have now found meanings for fractional, zero, and negative exponents consistent with the first law of indices. To give logical completeness, it is necessary to show that the meanings are consistent* with all the laws of indices, but we shall assume

* See Chrystal's *Algebra*, Fifth edition, Part I, p. 182.

this with the exception of one case which as an illustration we prove as follows:

To prove $a^{\frac{p}{q}} \cdot b^{\frac{p}{q}} = (ab)^{\frac{p}{q}}$, where p and q are positive integers, let

$$x = a^{\frac{p}{q}} \cdot b^{\frac{p}{q}}.$$

Then,
$$x^q = \left(a^{\frac{p}{q}} \cdot b^{\frac{p}{q}}\right)^q$$
$$= \left(a^{\frac{p}{q}}\right)^q \left(b^{\frac{p}{q}}\right)^q \qquad \text{(Law III, Art. 6.)}$$
$$= a^p b^p \quad \text{(Definition of fractional exponent.)}$$
$$= (ab)^p. \qquad \text{(Law III, Art. 6.)}$$

That is, x is the qth root of $(ab)^p$, or

$$x = (ab)^{\frac{p}{q}}.$$

Hence,
$$a^{\frac{p}{q}} \cdot b^{\frac{p}{q}} = (ab)^{\frac{p}{q}}.$$

We have now attached a meaning to the expression a^x provided x is any integer, or rational fraction, positive or negative. Moreover, we assume if a is positive and x is a rational number that the number described by a^x exists. It is possible also to give a meaning to a^x when x is not a rational number. However, that is beyond our present purpose. (See Art. 111.)

12. Radicals. As stated in Art. 7, the nth root of a, written $\sqrt[n]{a}$, has the same meaning as $a^{\frac{1}{n}}$. The number under the radical sign is often called the **radicand** and the number which indicates the root is called the **index**. Thus, in $\sqrt[n]{a}$, a is the radicand and n is the index.

The rules of operations with radicals may be obtained at once from the laws of indices. Thus,

I. $\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}.$

II. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}.$

EXERCISES

Obtain identical expressions free from negative exponents and simplify.

1. $5\,a^{-3}x^{3}$.

2. $\dfrac{5\,x^{3}}{a^{-3}}$.

3. $(9^{-1})^{\frac{1}{2}}$.

4. $(9)^{-\frac{1}{2}}$.

5. $(9^{-1})^{-\frac{1}{2}}$.

6. $(\frac{16}{25})^{\frac{1}{2}}$.

7. $(\frac{16}{25})^{-\frac{1}{2}}$.

8. $(\frac{16}{25})^{-1}$.

9. $(\frac{8}{27})^{-\frac{1}{3}}$.

10. $(0.49)^{\frac{1}{2}}$.

11. $(0.49)^{-\frac{1}{2}}$.

12. $\dfrac{2^{-4}\cdot 3^{2}\cdot 5^{-3}}{2^{-3}\cdot 3^{-2}\cdot 5}$.

13. $\dfrac{a^{0}x^{-1}}{ax^{-2}}$.

14. $\dfrac{2\,a^{-3}b^{2}c^{-4}}{5^{-1}a^{4}b^{-5}c^{-1}}$.

Perform the indicated operations and simplify.

15. $(a^{\frac{1}{2}})^{\frac{1}{2}}$.

16. $(a^{\frac{1}{3}}b^{2}c^{\frac{1}{2}})^{2}$.

17. $(16\,x^{4}y^{12})^{\frac{1}{4}}$.

18. $(a^{0}x^{-1}\cdot y^{\frac{1}{2}})^{-2}$.

19. $\left(\dfrac{a^{0.3}}{x^{0.7}}\right)^{0.2}$.

20. $\left(\dfrac{a^{5}x^{-5}}{x^{9}y^{\frac{3}{2}}}\right)^{-\frac{1}{3}}$.

21. $(-x^{6}y^{3}z^{-9})^{\frac{7}{3}}$.

22. $\left(\dfrac{p^{1.42}}{p}\right)^{-\frac{1}{3}}$.

23. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}})$.

24. $(a^{0}+b^{\frac{1}{2}})(a^{0}-b^{-\frac{1}{2}})$.

25. $(a^{\frac{4}{3}}+b^{\frac{4}{3}})(a^{\frac{4}{3}}-b^{\frac{4}{3}})$.

26. $x^{2}y\left(x^{\frac{1}{3}}y^{\frac{2}{3}}+\dfrac{1}{x^{\frac{1}{3}}y^{\frac{2}{3}}}\right)$.

27. $(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})(x^{-\frac{1}{2}}-y^{-\frac{1}{2}})$.

28. $(x^{0}+x^{\frac{1}{2}})(x^{0}-x^{\frac{1}{2}}+x)$.

29. $(a^{\frac{1}{3}}-a^{-\frac{1}{3}})(a^{\frac{2}{3}}+a^{-\frac{2}{3}}+1)$.

30. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})$.

31. $(a^{\frac{1}{3}}+b^{\frac{1}{3}})^{3}$.

32. $(x^{-2}+x^{-1}+x^{0})(x^{-2}-x^{-1}+x^{0})$.

33. $(p^{-2}-2\,p^{-1}+1)(p^{-1}-1)$.

34. $(x^{-\frac{3}{4}}+x^{-\frac{1}{2}})(x^{\frac{3}{4}}+x^{\frac{1}{2}})$.

35. $(6\,x^{2}y^{-1}-15\,yx^{-1})\div 3\,x^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

36. $(x^{-2}+x^{-1}-56)\div(x^{-1}-7)$.

37. $(m^{-3}+1)\div(m^{-1}+1)$.

38. $(x^{-4}-16\,y^{-4})\div(x^{-1}+2\,y^{-1})$.

39. $(a^{-10}-a^{-5}+1)\div(a^{-2}-a^{-1}+1)$.

40. $(x^{\frac{5}{2}}+x^{2}+x^{\frac{3}{2}}+x+x^{\frac{1}{2}}+x^{0})\div(x^{\frac{3}{2}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}})$.

41. $(a^{\frac{4}{3}}b^{-1}+4\,ab^{-\frac{1}{2}}-2\,a^{\frac{2}{3}}-12\,a^{\frac{1}{3}}b^{\frac{1}{2}}+9\,b)\div(a^{\frac{2}{3}}b^{-\frac{1}{2}}+2\,a^{\frac{1}{3}}-3\,b^{\frac{1}{2}})$.

42. $(a^{-\frac{1}{3}}x^{-\frac{1}{3}}+2+2\,a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}})\div(a^{-\frac{1}{3}}+x^{\frac{1}{3}})$.

Obtain identical expressions involving radicals instead of fractional exponents and simplify.

43. $x^{\frac{1}{2}}y^{\frac{2}{3}}$.

44. $a^{-\frac{1}{2}}b^{\frac{1}{4}}$.

45. $(a+b)^{\frac{5}{6}}$.

46. $(a-b)^{-\frac{5}{6}}$.

47. $(x^n \cdot y^{-m})^{\frac{1}{m}}$.

48. $a^{-3}b(ab^{-3})^{-\frac{1}{3}}$.

49. $7a^{\frac{3}{4}}a^{-\frac{4}{3}}$.

50. $[x^3(y^2z)^{\frac{1}{3}}]^{\frac{3}{2}}$.

51. $[(a^{-\frac{1}{3}} \cdot x^{-\frac{2}{3}}z^{-1})^{\frac{1}{3}}]^{\frac{3}{4}}$.

Obtain identical expressions involving fractional exponents instead of radicals and simplify.

52. $\sqrt{x^5y^2z^3}$.

53. $\sqrt[3]{(a^{-3}x^{-2})^4}$.

54. $\sqrt[3]{a^4\sqrt{a^6x^3}}$.

55. $\sqrt{x^3z^5} + \sqrt[3]{x^4z^6} - \sqrt[4]{x^6z^{10}}$.

56. $\sqrt[3]{\dfrac{a^2b^4}{x^3y^0}}$.

57. $\sqrt{\dfrac{32\,x^5y}{2^3}\sqrt{\dfrac{2^4}{x^4y^2}}}$.

58. $\dfrac{1}{\sqrt[3]{\dfrac{1}{\sqrt[4]{8}}}}$.

59. $\sqrt[n]{ax^2y^n}$.

60. $\sqrt[n]{\dfrac{x^{n-1} \cdot y^m}{z^{2n}}}$.

Simplify the following.

61. $\left[a^{\frac{1}{n-1}} \cdot a^{\frac{1}{n+1}}\right]^{\frac{1}{n}}$.

62. $\left[(a^{p+q})^{p-q} \cdot (a^{2q})^{\frac{1}{q}}\right]^{\frac{1}{2p}}$.

63. $\sqrt{x\sqrt{x\sqrt{x}}}$.

64. $\left(a^{2n} \cdot a^n \cdot b^{\frac{1}{3n}}\right)^{3n}$.

65. $\left[x^{\frac{2}{3}}\sqrt{\left(\dfrac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^7}\right]^{\frac{3}{21}}$.

66. If $x^a = a^x$, show that $x \cdot \left(\dfrac{x}{a}\right)^{\frac{x}{a}} = x^{\frac{x}{a}}$.

67. The number of revolutions per minute of a water-wheel is given by the formula

$$n = 53.25\,\frac{F^{\frac{5}{4}}}{H^{\frac{1}{2}}},$$

where F is the fall of the water in feet, H the horse power. Calculate n when $F = 16$, $H = 100$.

68. We may write 0.000,016 in the form $16 \cdot 10^{-6}$. Write the following numbers in briefer form by the use of negative exponents: 0.000,001, 0.000,000,003, 0.000,001,5.

69. Calculate the following differences.

$[(\frac{1}{2})^{-2} - (2)^{-2}]$; $[-(2)^{-2} - (-2)^{-2}]$; $[(\frac{1}{2})^{-2} - 2^2]$; $[(\frac{1}{2})^{-2} - (-2)^{-2}]$.

70. If $3^{\frac{1}{2}}$ and $2^{\frac{1}{2}}$ are substituted for x in the expression

$$x^5 - x^4 - 5x^3 + 5x^2 + 6x,$$

show that the results reduce to the same number.

71. Two spherical particles each one gram in mass whose centers are one centimeter apart attract each other with a force of 0.000,000,066,6 dyne. Express this number as an integer multiplied by 10 with an exponent.

72. By the use of negative exponents express a micron as a part of a meter. (A micron is one millionth of a meter.)

73. The numbers 6867, 5896, 4861, 3934 each multiplied by 10^{-10} give the wave lengths in meters of deep red, yellow, blue, and ultra violet light respectively. Express each wave length in microns.

74. Some authorities say that the mass of a hydrogen atom is 10^{-25} gram. How would this number be written in ordinary decimal notation?

CHAPTER II

ALGEBRAIC REDUCTIONS

13. Algebraic expressions. In algebra, an **expression** is a symbol or combination of symbols that represents a number. Thus, $x^2 + y^2 - 25$ and $\frac{1}{2}gt^2 + vt$ are expressions, if x, y, g, t, v, represent numbers. If $x = 4$ and $y = 6$, the first takes the value 27. If $g = 32.2$, $t = 10$, and $v = 5$, the second has the value 1660. For different values of the letters, an expression represents in general different numbers. On the other hand, the same numbers may be represented by many different expressions. For example, $x^2 - 4$ and $(x-2)(x+2)$ represent the same number. Expressions which are equal for all values of the symbols for which the expressions are defined * are called **identical expressions.** The statement that two identical expressions are equal is called an **identity.** Thus, $x^2 - 4$ and $(x-2)(x+2)$ represent the same number no matter what value is assigned to x, and the statement

$$x^2 - 4 = (x-2)(x+2)$$

is an identity.

Two expressions may be equal without being identical. Thus, for $x = 2$ or -5, the expressions $x^2 + 2x - 1$ and $9 - x$ are equal, but they are equal for no other values of x, and hence are not identical. Frequently, in problems with which we shall deal, the work is made easier by replacing expressions by identical but simpler expressions.

Exercise. Which of the following are identical expressions?

$$x^2 - y^2,\ (x-y)(x+y),\ 4x - y,\ \frac{12x^2 - 3xy}{3x},\ 4x(x+y),\ 4x^2 + 4xy.$$

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* This statement implies that we may not assign values to the letters which make the members meaningless. Thus, $\frac{1}{1-x} = 1 + \frac{x}{1-x}$ is excluded when $x = 1$.

14. Removal of parentheses. A single pair of parentheses can always be removed from an algebraic expression. However, if the sign before it is $+$, no change is necessary, but if $-$, the signs of the terms within it must be changed (Art. 5, VII, VIII).

Expressions often occur with more than one pair of parentheses. When one pair occurs within another pair, other symbols besides ( ) are used as follows: [ ] called *brackets*, { } called *braces*, and $\overline{\quad}$ called the *vinculum*. All parentheses may be removed by first removing the innermost pair according to the rule for a single pair; next, the innermost pair of all that remain, and so on.

EXERCISES

Remove the parentheses from the expressions of the first four exercises.

1. $8a+(5b-a)-(2a-7b)$.

2. $a-(b-\overline{c+d}+e)$.

3. $2x-[3x-\{x-(2x-\overline{3x+4})\}-(5x-2)]$.

4. $3x-\{y-[y-(x+y)-\{-y-(y-\overline{x-y})\}]\}$.

5. Find the numerical value of the expression in Exercise 4 when $x=2$, $y=-1$.

6. Without changing the value of the expression

$$4c+3b+4a-5c^2+4d^3,$$

write it with the last three terms in a parenthesis preceded by a minus sign.

7. Fill out the last pair of parentheses in $a+b-(c+d)=a-d-(\quad)$.

8. Find the numerical value of

$$3x-(5x+[-4x-\overline{y-x}])-(-x-3y),$$

when $x=2$, $y=4$.

9. Annex a pair of parentheses preceded by a minus sign to $6a+5b-(a+b)$ so that the resulting expression equals $a+b$.

10. Simplify $\{x(x+y)-y(x-y)\}\{x(x-y)+y(x+y)\}$.

15. Complex fractions. A **complex fraction** is one which has a fraction in the numerator or denominator or in both numerator and denominator. The rules for the simplification of arithmetical fractions apply to algebraic fractions no matter how complicated the numerator or denominator may be.

The main principle is that (Art. 5, IX) the value of a fraction is not changed by multiplying numerator and denominator by the same number.

As illustrated by the following exercises on complex fractions, a simplification is often brought about if we select for the number by which to multiply, the lowest common denominator of the fractions which are in the numerator and denominator of the complex fraction.

EXERCISES

Simplify :

1. $\dfrac{x}{x-\dfrac{xy}{z}}$.

2. $\dfrac{1-\dfrac{1}{x^2}}{1-\dfrac{1}{x}}$.

3. $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$.

4. $\dfrac{1+\dfrac{1}{2m}}{m-\dfrac{1}{4m}}$.

5. $\dfrac{\dfrac{1}{a}-a}{1+a}$.

6. $\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}+2}$.

7. $\dfrac{x-\dfrac{1}{x^2}}{1-\dfrac{1}{x}}$.

8. $\dfrac{\dfrac{x}{y}-\dfrac{y}{z}}{\dfrac{x}{y}+\dfrac{y}{z}}$.

9. $\dfrac{\dfrac{m}{n}-\dfrac{n}{m}}{\dfrac{(m+n)^2}{mn}-4}$.

10. $1-\dfrac{1}{1+\dfrac{1}{x}}$.

11. $\dfrac{\dfrac{x^2}{y^2}+\dfrac{1}{x}}{\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}}$.

12. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$.

13. $\dfrac{\dfrac{x+y}{y}-\dfrac{x+y}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}$.

14. $\dfrac{y-x+\dfrac{x^2}{y}}{\dfrac{x}{y^2}+\dfrac{y}{x^2}}$.

15. $\dfrac{x-2+\dfrac{1}{x+2}}{x+2+\dfrac{1}{x-2}}$.

16. $\dfrac{1}{1-\dfrac{1}{1+x}}+\dfrac{1}{1-\dfrac{1}{1-x}}$.

17. $\dfrac{1-\dfrac{1}{x}}{\dfrac{x}{\dfrac{1}{x^2}}}$.

18. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}$.

19. $\dfrac{x^2-\dfrac{1}{x}}{x+\dfrac{1}{x}+1}$.

20. $\dfrac{\dfrac{n}{n-1}+\dfrac{n}{n+1}}{\dfrac{n}{n-1}-\dfrac{n}{n+1}}$.

21. $\dfrac{\dfrac{1}{x}-\dfrac{1}{x+h}}{h}$.

22. $\dfrac{\dfrac{x+h}{x+h-1}-\dfrac{x}{x-1}}{h}$.

23. $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{(x+h)^2}}{h}$.

24. $\dfrac{\dfrac{1}{r}-\dfrac{1}{s}}{r^2-s^2}$.

25. $\dfrac{\dfrac{E}{t}}{\dfrac{3r}{t}+R}$.

26. $\dfrac{1+\dfrac{7R}{3p}}{\dfrac{R}{6p}}$.

Express the following as complex fractions and then simplify.

27. $\dfrac{ax^{-2}}{a^2x^{-3}}$.

28. $\dfrac{a^{-2}b^3}{a^{-4}b^2}$.

29. $\dfrac{a-x^{-1}}{ax^{-1}-x^{-2}}$.

30. $\dfrac{a^{-2}-b^{-2}}{a^{-1}-b^{-1}}$.

31. $\dfrac{a^{-1}b^2x^{-2}-2\,a^{-1}b^2y^3}{a^{-5}x^{-3}-2\,a^{-5}y^{-3}}$.

32. $\dfrac{ab^{-2}+ba^{-2}}{a^{-2}-a^{-1}b^{-2}+b^{-2}}$.

33. $\dfrac{x+(a^2-x^2)^{\frac{1}{2}}}{x+x(a^2-x^2)^{-\frac{1}{2}}}$.

34. $\dfrac{(a^2-x^2)^{\frac{1}{2}}}{x(a^2-x^2)^{-\frac{1}{2}}}$.

16. Factoring. Certain useful methods of resolving algebraic expressions into their factors are illustrated in the problems which are worked out in the following list.

EXERCISES

Separate each of the following into two factors.

1. $6\,a^2bx^3 + 2\,ab^2x^4 + 4\,abx^5$.

2. $10\,m^4n^2 - 15\,m^3n^3$.

Solution: $2\,abx^3(3\,a + bx + 2\,x^2)$.

3. $4\,a^2bc + 8\,a^2b^2c^2 - 16\,a^3b^2c$.

4. $14\,xyz - 7\,x^2yz + 28\,xy^2z$.

5. $a^2 + ab + ac + bc$.

Solution: $a^2 + ab + ac + bc = a(a + b) + c(a + b) = (a + c)(a + b)$.

6. $x^2 + 5\,x + xy + 5\,y$.

7. $ax - 3\,by + bx - 3\,ay$.

8. $xy + x - 2\,y - 2$.

9. $m^2 - n^2 - (m - n)^2$.

10. $x^2 - 2\,x - 8$.

Solution: The result will be of the form $(x + a)(x + b)$. This expanded is $x^2 + (a + b)x + ab$. Comparing with $x^2 - 2\,x - 8$, we see that we can find factors of the given expression if we can find two numbers a and b such that $a + b = -2$, $ab = -8$, or $a = -4$, $b = 2$. Hence, $x - 4$ and $x + 2$ are factors of $x^2 - 2\,x - 8$.

11. $x^2 - 3\,x + 2$.

12. $2\,x^2 - 6\,x + 4$.

13. $y^2 - 7\,y + 12$.

14. $a^2 + 7\,a - 30$.

15. $s^2 - 4\,s - 60$.

16. $a^2b^2 - 7\,ab + 10$.

17. $8\,x^2 - 18\,x + 9$.

18. $a^3 - b^3$.

Solution: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

19. $a^3 - 27$.

20. $m^6 - 1$.

21. $16\,a^4 - b^4$.

22. $a^4 - 81$.

23. $r^6 - s^6$.

24. $1 - b^6$.

25. $81\,s^4 - 49\,t^4$.

26. $a^3 - (a - b)^3$.

27. $x^2y^2 - 9\,a^2b^2$.

28. $x^2 + (b - a)x - ab$.

29. $(3\,x - 4)^2 - (2\,x - 2)^2$.

30. $a^2 + 2\,ab + b^2 - c^2$.

31. $x^2 + y^2 - 9\,x^2y^2 + 2\,xy$.

32. $64\,a^3 - 27\,y^3$.

33. $1 - 343\,x^3$.

34. $27\,a^3 - 8\,x^3$.

35. $a^3 + b^3$.

36. $27 + 8\,x^3$.

37. $64\,x^6 + 125\,y^3$.

38. $x^3y^3 + 216$.

39. $a^5 - b^5$.

40. $(a + b + c)^2 - (a - b - c)^2$.

41. $x^4 + x^2y^2 + y^4$.

42. $x^9 - 8\,x^3 - x^6 + 8$.

43. $1 + a^2 + a^4$.

44. $16\,x^4 + 4\,a^2b^4x^2 + a^4b^8$.

45. $a^4 + 4$.

46. $(3\,m + 2)^3 + (2\,m + 1)^3$.

17. Radicals. A **rational** number is one which can be represented as the quotient of two integers. In operations with radi-

cals we deal with **irrational** numbers; that is, with numbers which cannot be expressed as the quotient of two integers.* Any irrational number can always be inclosed between two rational numbers which differ from one another by as small a number as we please.

Thus, we may write,

$$\begin{aligned} 1 &< \sqrt{2} < 2, \\ 1.4 &< \sqrt{2} < 1.5, \\ 1.41 &< \sqrt{2} < 1.42, \\ 1.414 &< \sqrt{2} < 1.415, \\ &\cdots \end{aligned}$$

Either of the sequences of numbers in the two outer columns determines the square root of 2 in the same way that the sequence .3, .33, .333, .3333, ··· determines the fraction $\frac{1}{3}$. Any irrational real number may be determined in this way by a sequence of rational numbers. Besides those numbers in which a radical sign is used to express their exact values, there are many others which belong to the class of irrational numbers. For example, the number π is an irrational real number determined by the sequence 3, 3.1, 3.14, 3.141, 3.1415, ···.

Rules of operation with expressions involving radicals are given in connection with the treatment of fractional exponents (Art. 12).

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*To show that $\sqrt{2}$ cannot be expressed as the quotient of two integers, suppose it possible that $\sqrt{2} = \frac{m}{n}$, where $\frac{m}{n}$ is a rational fraction in its lowest terms. At least one of the numbers m or n is odd. Clearing of fractions and squaring both sides, we get $2n^2 = m^2$.

From this equation m^2 is an even number, hence m is an even number. If m is even, m^2 contains the factor 4. Hence n^2 is an even number, and n is itself even. This is contrary to the hypothesis that $\frac{m}{n}$ is a fraction in its lowest terms.

This proof is found in Euclid, and is supposed to be due to a much earlier mathematician than Euclid.

EXERCISES

Introduce the coefficients * of the following seven radicals under the radical sign.

1. $2\sqrt{3}$. SOLUTION: $2\sqrt{3} = \sqrt{4} \cdot \sqrt{3} = \sqrt{12}$.

2. $3\sqrt{11}$. **3.** $2\sqrt[3]{5}$. **4.** $2\sqrt[4]{7}$.

5. $2\sqrt[3]{13}$. **6.** $7\sqrt[3]{6}$. **7.** $4\sqrt[3]{3}$.

8. Perform operations in Exercises 1–7, using fractional exponents instead of radicals.

Change the following seven expressions into equivalent expressions having no fractions under the radical sign.

9. $\sqrt{\frac{2}{3}}$. SOLUTION: $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\sqrt{6}$.

10. $\sqrt[3]{\frac{1}{5}}$. **11.** $\sqrt[4]{\frac{2}{5}}$. **12.** $\sqrt{\frac{4}{7}}$.

13. $3\sqrt[5]{\frac{1}{3}}$. **14.** $6\sqrt[4]{\frac{7}{8}}$. **15.** $7\sqrt{\frac{6}{7}}$.

16. Change the radicals in Exercises 9–15 to fractional exponents and reduce each to a fraction whose denominator is free from fractional exponents.

Change the following eight radicals into equal expressions with as small a positive integer as possible under the radical sign.

17. $\sqrt{8}$. SOLUTION: $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.

18. $\sqrt[3]{32}$. **19.** $\sqrt{18}$. **20.** $\sqrt{125}$.

21. $\sqrt[4]{48}$. **22.** $\sqrt[3]{81}$. **23.** $\sqrt{800}$.

24. $\sqrt[6]{2187}$.

25. Perform operations in Exercises 17–24, using fractional exponents.

Change the following into equivalent expressions whose indices are the smallest possible positive integers.

26. $\sqrt[4]{4}$.

SOLUTION: $\sqrt[4]{4} = \sqrt{\sqrt{4}} = \sqrt{2}$.

27. $\sqrt[6]{8000}$. **28.** $\sqrt[6]{81}$. **29.** $\sqrt[4]{169}$. **30.** $\sqrt[6]{343}$.

31. $\sqrt[8]{64}$.

32. Perform operations in Exercises 26–31, using fractional exponents.

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* By the coefficient of a radical, we mean the number by which the radical is multiplied. In general, the coefficient of any symbol is the expression by which the symbol is multiplied.

Change the following fractions into equivalent fractions, having no radicals in the denominators.

33. $\dfrac{1}{1+\sqrt{2}}$.

SOLUTION: $\dfrac{1}{1+\sqrt{2}} = \dfrac{1}{1+\sqrt{2}} \cdot \dfrac{1-\sqrt{2}}{1-\sqrt{2}} = \sqrt{2}-1.$

34. $\dfrac{1}{\sqrt{5}}$. **35.** $\dfrac{2}{\sqrt[3]{2}}$. **36.** $\dfrac{1}{1-\sqrt{3}}$. **37.** $\dfrac{5}{\sqrt{8}+3}$.

38. $\dfrac{3}{\sqrt{5}+\sqrt{3}}$. **39.** $\dfrac{5}{\sqrt{5}-\sqrt{\frac{1}{2}}}$. **40.** $\dfrac{4}{\sqrt{2}-\sqrt{3}}$.

41. Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$, evaluate the expressions in Exercises 33, 34, 36, 38 (to four significant * figures).

18. Reduction of expressions containing radicals to the simplest form. An expression containing radicals is said to be in its simplest form with regard to the radicals when, (*a*) there are no fractions under the radical signs; (*b*) the radicand contains no factor raised to a power whose exponent equals the index of the root; (*c*) the indices of the radicals are positive integers and as small as possible; (*d*) there are no radicals in the denominator.

EXERCISES

Reduce to the simplest form.

1. $\dfrac{\sqrt[6]{4}+\sqrt{\frac{4}{3}}}{\sqrt{3}}$.

SOLUTION: $\dfrac{\sqrt[6]{4}+\sqrt{\frac{4}{3}}}{\sqrt{3}} = \dfrac{\sqrt[3]{\sqrt{4}+2\sqrt{\frac{1}{3}}}}{\sqrt{3}} = \dfrac{\sqrt[3]{2}+\frac{2}{3}\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt[3]{2}+\frac{2}{3}\sqrt{3}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt[6]{108}+2}{3}.$

2. $\sqrt[3]{1\frac{2}{5}}$. **3.** $\sqrt{\frac{5}{7}}$. **4.** $\dfrac{3}{\sqrt[3]{11}}$. **5.** $\sqrt{1215}$. **6.** $\sqrt[10]{32}$.

* In giving a result such as 2.2361 to four significant figures, we give 2.236. In giving the same result to three significant figures, we give 2.24 rather than 2.23, for 2.24 differs less from 2.236 than 2.23 differs from 2.236. In fact, it is usually desirable in giving any number of figures of an approximate result, to find whether the next figure beyond those to be retained in the result is less or greater than 5; for, if we should obtain a result 2.23 and know that the next figure > 5, the result should be given to three significant figures as 2.24.

7. $\dfrac{7-\sqrt{3}}{7+\sqrt{3}}$. **8.** $\dfrac{3\sqrt{\frac{1}{3}}}{2+5\sqrt{\frac{1}{3}}}$. **9.** $\dfrac{\sqrt{2}+\sqrt{5}-\sqrt{3}}{\sqrt{2}-\sqrt{5}+\sqrt{3}}$. **10.** $\dfrac{\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}}{\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{2}}}$.

Express in terms of radicals in simplest form.

11. $\dfrac{3^{\frac{1}{2}}+2^{\frac{1}{2}}}{3^{\frac{1}{2}}-2^{\frac{1}{2}}}$. **12.** $\dfrac{2(5)^{\frac{1}{2}}+(3)^{\frac{1}{2}}}{3(5)^{\frac{1}{2}}-2(3)^{\frac{1}{2}}}$. **13.** $\dfrac{a^{\frac{3}{4}}-a^{-\frac{3}{4}}}{a^{\frac{1}{4}}-a^{-\frac{1}{4}}}$. **14.** $\dfrac{x-1}{x^{\frac{1}{3}}-1}-\dfrac{x^{\frac{2}{3}}-1}{x^{\frac{1}{3}}+1}$.

19. Addition and subtraction of radicals. Radicals having the same index and same number under the radical sign are called *similar* radicals. Similar radicals can be combined in addition and subtraction.

EXERCISES

Perform the operations indicated.

1. $2\sqrt{6}+3\sqrt{6}+\sqrt{24}-2\sqrt{54}$.

SOLUTION: $2\sqrt{6}+3\sqrt{6}+\sqrt{24}-2\sqrt{54}=2\sqrt{6}+3\sqrt{6}+2\sqrt{6}-6\sqrt{6}=\sqrt{6}$.

2. $\sqrt{3}+4\sqrt{3}-2\sqrt{3}$. **3.** $\sqrt{24}+\sqrt{54}-\sqrt{96}$.

4. $\sqrt{48\,xy^2}+y\sqrt{75\,x}+\sqrt{3\,x(x-9\,y)^2}$.

5. $\sqrt{3}+\sqrt{27}+3\sqrt{27}-\sqrt{75}+\sqrt[4]{9}$. **6.** $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{18}$.

7. $b\sqrt{a}+x\sqrt{a}-y\sqrt{a^3}$. **8.** $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{2}}$.

9. $2\sqrt{\frac{5}{8}}+\frac{1}{6}\sqrt{60}+\sqrt{15}+\sqrt{\frac{3}{5}}$. **10.** $6\sqrt[6]{4\,x^2}+2\sqrt[3]{2\,x}+\sqrt[6]{8\,x^3}$.

11. $5\sqrt[3]{a}+3\sqrt{a}+\sqrt{a}+5\sqrt{a}-2\sqrt{a}-7\sqrt[3]{a}$.

12. $20(3)^{\frac{1}{2}}-4(3)^{\frac{1}{2}}$. **13.** $(243)^{\frac{1}{2}}+(27)^{\frac{1}{2}}+(48)^{\frac{1}{2}}-3^{\frac{1}{2}}$.

14. $(405)^{\frac{1}{2}}-(180)^{\frac{1}{2}}$. **15.** $(135)^{\frac{1}{3}}-(40)^{\frac{1}{3}}$.

20. Multiplication and division of radicals. In Art. 12 we have seen that $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$. In multiplying or dividing radicals with different indices, the radicals should first be reduced to the same index.

EXERCISES

Perform the indicated operations.

1. $\sqrt[3]{3}\cdot\sqrt{5}$.

SOLUTION: $\sqrt[3]{3}=\sqrt[3]{\sqrt{9}}=\sqrt[6]{9}$.

$\sqrt{5}=\sqrt{\sqrt[3]{125}}=\sqrt[6]{125}$.

Hence, $\sqrt[3]{3}\cdot\sqrt{5}=\sqrt[6]{9}\cdot\sqrt[6]{125}=\sqrt[6]{1125}$.

2. $\sqrt{3}\cdot\sqrt[3]{5}$. 3. $\sqrt{2a}\cdot\sqrt[3]{3a^2}$. 4. $5\sqrt{3}\cdot 5\sqrt[3]{3}\cdot\sqrt{2}$. 5. $4\sqrt[3]{12}\cdot 3\sqrt[3]{4}$.

6. $\sqrt[5]{7}\cdot\sqrt[5]{11}$. 7. $3\sqrt[3]{a}\cdot 4\sqrt[4]{b}$. 8. $(\sqrt{7}-1)(\sqrt{7}+1)$.

9. $(2\sqrt{3}+3\sqrt{2})(3\sqrt{3}-\sqrt{2})$. 10. $(11\sqrt{2}-4\sqrt{15})(\sqrt{6}+\sqrt{5})$.

11. $(\sqrt[3]{m}+\sqrt[3]{n})^3$. 12. $(\sqrt[3]{12-\sqrt{19}})(\sqrt[3]{12+\sqrt{19}})$.

13. $(3\sqrt{1+a}-4\sqrt{a})(\sqrt{1+a}+2\sqrt{a})$. 14. $(x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$.

15. $\sqrt[3]{x}\cdot\sqrt{y}\cdot\sqrt[4]{z}$. 16. $\sqrt[2n]{a}\cdot\sqrt[n]{a^2}\cdot\sqrt[3n]{a^3}$.

17. $\dfrac{70\sqrt[3]{9}}{7\sqrt[3]{18}}$. 18. $\dfrac{\sqrt[6]{72}}{\sqrt{2}}$.

19. $\sqrt{2}\cdot\sqrt[3]{4}\cdot\sqrt[4]{3}$. 20. $\dfrac{4\sqrt[3]{9}}{2\sqrt{3}}$.

21. $\sqrt[4]{\dfrac{x}{y}}\div\sqrt[4]{\dfrac{y}{x}}$. 22. $\dfrac{5}{\sqrt{3}}\div\sqrt{\dfrac{3}{5}}$.

23. $a\div\sqrt{\dfrac{a}{b}}$. 24. $\dfrac{\sqrt{28x}}{\sqrt[4]{42x}}$.

25. $\dfrac{(18)^{\frac{1}{3}}}{3^{\frac{1}{3}}}$. 26. $\dfrac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}$.

27. $\dfrac{7^{\frac{1}{3}}}{7^{\frac{1}{2}}}$. 28. $\sqrt[4]{5ab}\div\sqrt[8]{125b^2a}$.

29. $\sqrt[3]{12a^2}\div\sqrt{8a^3}$. 30. $\sqrt{6ax^2}\div\sqrt[5]{9a^2x^4}$.

21. Evaluation of formulas.

1. The speed v in feet per second of a projectile of weight w pounds, and diameter d inches is given by

$$\frac{1}{v}=\frac{1}{v_0}+\frac{td^2}{7000\,w},$$

where v_0* is muzzle speed with which the shell is projected, and t is the number of seconds after leaving the muzzle. Find v when $v_0=2800$, $t=4$, $d=14$, $w=1400$.

2. $H=\dfrac{v}{825}\left(T-\dfrac{wv^2}{g}\right)$.

Find H when $T=400$, $w=0.7$, $g=32.2$, and $v=80$.

3. $T=\dfrac{\left(P+\dfrac{a}{v^2}\right)(v-b)}{R}$.

Find T when $a=1222$, $b=0.01869$, $R=35.24$, $v=1.550$, $P=11000$.

* A letter with a subscript, say a_r, is read, "a sub r."

4. $w = w' - s(s_0 - s_1)$.

Find w when

$$w' = 10.700,$$
$$s = 0.0050,$$
$$s_0 = 8.5,$$
$$s_1 = 10.9.$$

5. The specific gravity S of a floating body is given by the expression

$$S = \frac{w_1}{w_1 + (w_2 - w_3)}$$

where w_1 is the weight of the body in air, w_2 is the weight of a sinker in water, and w_3 is the weight in water of the body with sinker attached.

Determine the specific gravity of a body when by physical measurements it is found that

$$w_1 = 16.50,$$
$$w_2 = 182.3,$$
$$w_3 = 176.5.$$

6. One cubic centimeter of mercury at $x°$ centigrade increases in volume when heated to $y°$ by an amount given by the following formula:

$$\frac{\frac{A(y-x)}{100}}{1 + \frac{Ax}{100}},$$

where $A = 0.018$. Find the increase in volume when the temperature is raised from 12° to 126°.

7. The sag of an overhead trolley wire in an electric tramway is given by the formula

$$d = \sqrt{\frac{3L(l-L)}{8}},$$

where d is the number of feet in the sag, l is the number of feet of wire between poles, L is the number of feet from pole to pole. Find the sag when poles are 100 feet apart and length of wire is 100.1 feet.

8. The time in seconds required for the discharge of water from one vessel to another through an orifice in the side is

$$t = \frac{0.116 \cdot A \cdot B \cdot (F^{\frac{1}{2}} - f^{\frac{1}{2}})}{(A + B) \cdot a},$$

where F and f are the differences in the heights of water in the two vessels at the beginning and end respectively of the discharge, a is the area of the orifice, A is the area of a horizontal section of the discharging vessel, and B is that of the receiving vessel (measurements in inches).

Find t when $F = 128$, $f = 92$, $A = 96$, $B = 60$, and the orifice is a circle one inch in diameter.

9. The area in square feet of the top of a well-designed chimney is given by the formula

$$A = 0.03\frac{Q}{\sqrt{h}},$$

where Q is the quantity of coal in pounds used per hour and h is the height of the chimney. What should be the area of the top of a chimney 150 feet high which is connected with a furnace using 11,000 pounds of coal per hour?

10. To correct a barometer reading for temperature the following amount is subtracted from the reading:

$$B\frac{m(t-32)-s(t-62)}{1+m(t-32)},$$

where B is the barometer reading in inches, t the temperature in degrees Fahrenheit, $m = 0.00010$, $s = 0.00001$. What is the corrected reading of the barometer when the temperature is 75° and the barometer reads 29.95?

11. The quantity of water in cubic feet per second flowing through a rectangular weir is given by the formula

$$Q = 3.33 \cdot [L - 2h]h^{\frac{3}{2}},$$

where h is the depth of water over the sill of the weir in feet, and L the length of the sill.

Find Q, where $L = 26$, $h = 1.6$.

12. Let P be the day of the month, q the number of the month in the year, counting January and February as the 13th and 14th months of the preceding year, N the year, and $n = \left[\frac{N}{100}\right] - \left[\frac{N}{400}\right] - 2$.

If

$$P + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - n$$

be divided by 7, the remainder will be the day of the week of a given date where Sunday counts as the first day. The expressions in brackets mean the largest integer contained in the inclosed number. Verify this formula for the present date.

13. Three equal uniform rods of weight w and the length l are jointed together to form a triangle ABC; this triangle is hung up by the joint A, and a weight W is attached to B and C by two strings of length $\frac{l}{\sqrt{2}}$. The compression in BC is given by

$$x = W\frac{\sqrt{3}+1}{2\sqrt{3}} + \frac{w}{\sqrt{3}}.$$

Obtain x correct to nearest unit when $W = 1000$, $w = 500$.

14. The area of a triangle whose sides are a, b, c is given by the formula

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$. Calculate from this formula the area of a triangle whose sides are 5, 12, and 13 inches.

15. The formula for the horse power H.P. of an automobile engine is by

$$\text{H.P.} = \frac{Planc}{(24)(33000)},$$

where P is the pressure in pounds per square inch, l is the length of stroke of the piston in inches, a is the area of the end of the piston in square inches, n is the number of revolutions of the fly-wheel per minute, c is the number of cylinders in the engine. How many H.P. are developed by a six-cylinder engine if P is 72, l is 5, a is 15.56, and n is 1200 ?

22. Imaginary numbers. The square root of a negative number is an example of an imaginary number mentioned in Art. 2. To perform operations with square roots of negative numbers, replace any such number, say $\sqrt{-a}$, by $i\sqrt{a}$, and operate with i as with any other letter; but replace i^2 in any expression by -1. This method of operation is justified in Chapter XII.

EXERCISES

Perform indicated operations and simplify when possible by making $i^2 = -1$.

1. $(1+i)(2-i)$.

Solution:
$$\begin{aligned}(1+i)(2-i) &= 2 + i - i^2,\\ &= 2 + i + 1, \text{ since } i^2 = -1,\\ &= 3 + i.\end{aligned}$$

2. $(2-i)(2+i)$.

3. $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$.

4. $(-1+\sqrt{-3})^2$.

5. $(x+ai)(x-ai)$.

6. $\frac{i^3+1}{i+1}$.

7. $(2-\sqrt{-3})(2+\sqrt{-3})$.

8. $(24+7i)(24-7i)$.

9. $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

10. $(1-i^2)(1+i^2)$.

11. $(4+3i)^2 + (4-3i)^2$.

12. $(a+bi)^3 + (a-bi)^3$.

13. $(3+4i)^4 + (3-4i)^4$.

14. $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 1$.

15. $(x-a-bi)(x-a+bi)$.

CHAPTER III

VARIABLES AND FUNCTIONS

23. Constants and variables. A **constant** is a symbol which represents the same number throughout a discussion. A **variable** is a symbol which may represent different numbers in the discussion or problem into which it enters. Many mathematical expressions contain both variables and constants. Except in certain geometrical and physical formulas it is customary to use the letters a, b, c, $\cdots$ from the beginning of the alphabet for constants and the letters $\cdots$ x, y, z, from the end of the alphabet for variables.

Exercise. If A and B are two points in a plane and a point P moves in a circle about A as a center, which of the distances PA, PB is constant? which variable?

24. Definition of a function. Many problems in mathematics, physics, engineering, and chemistry involve two variables which are so related that, a value of one being given, the other can be found. The relation between the variables may be exhibited in various ways. Sometimes the values of the variables are arranged in the form of a table. For example, a life insurance agent refers to a table to find the premium corresponding to a given age. Here the two variables are "premium" and "age."

In algebra one variable may be connected with another in an equation or one variable may be an algebraic expression containing the other. In the equation $3x - 5y = 4$, if a value be given to x the corresponding value of y can be found. Thus if $x = 0$, $y = -\frac{4}{5}$; if $x = 1$, $y = -\frac{1}{5}$, and so on.

In evaluating the expression $x^2 + x + 1$, we find that $x^2 + x + 1 = 1$ when $x = 0$, $x^2 + x + 1 = 3$ when $x = 1$ and so on. Fixing the value of x in the first illustration fixes the value of y; in the second illustration fixing the value of x fixes the value of $x^2 + x + 1$.

Definition. If two variables are so related that when a value of one is given, a corresponding value of the other is determined, the second variable is called a **function** of the first.

Thus in the equation $3x - 5y = 4$, y is a function of x. The expression $x^2 + x + 1$, and in general any expression containing x, is a function of x. We may therefore and shall speak of a "function of x" instead of "an expression involving the variable x."

25. Functional notation. The symbols $f(x)$, $F(x)$, $\phi(x)$, $\cdots$ are used to represent functions of the variable x. They are read, "f" function of x, "F" function of x, etc. If, in a discussion, $f(x)$ is a function $x^2 + 3x - 1$, then $f(a)$ is $a^2 + 3a - 1$ and $f(x + h)$ is $(x + h)^2 + 3(x + h) - 1$. Similarly, if

$$\phi(x) = x^2 + 5x,$$

then $$\phi(2) = 2^2 + 5 \cdot 2,$$

and $$\phi(1 + y) = (1 + y)^2 + 5(1 + y).$$

These illustrations bring out an important point in the functional notation, viz.: If the same functional symbol, say $\phi(\ )$, be used more than once in any discussion, it stands in each case for the same operation or series of operations on the number or expression contained in the parentheses of the functional symbol. This notation is very convenient because it enables us to indicate the value of the function for any values of the variable on which it depends.

EXERCISES

1. The fact that the area of a circle may be calculated from a given radius may be expressed in the functional notation by $A = f(r)$. Give the particular form of $f(r)$ in this illustration.

2. The volume, V, of a cube is a function of the edge x. Express this fact in functional notation.

3. The hypotenuse of a right angled triangle is of length 10. Express the length of one leg y of the triangle as a function of the length of the other leg x.

4. Boyle's law says that $P \cdot V = C$, where P is the pressure of a gas, V its volume and C a constant for a given temperature. Express P as a function of V. Write in symbolic language the fact that C depends on the temperature T.

5. If $f(x)$ is $x^2 + x + 1$, find $f(a), f(o), f(1), f(-1), f(-10), f(a+1)$.

6. Given $\phi(x) = \dfrac{x+1}{x-1}$. Find $\phi(2), \phi(0), \phi(-1), \phi(1-\sqrt{2}), \phi(a+b)$.

7. Given $F(y) = \dfrac{y^2+1}{y^2-1}$. Find $F(o)$, $F\left(\dfrac{1}{x}\right)$, $F(\sqrt{x})$.

8. If $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$, show that $f(1) = f(-1) = f(3) = f(-\frac{1}{2})$.

9. Given $\phi(x) = x^2 + x + 1$. Find $\phi(x^2)$, $\phi(x+1)$, $\phi(\phi(x))$.

10. Given $y = f(x) = \dfrac{x+1}{2x-1}$. Show that $f(y)$ reduces to x.

11. Given $s = \phi(t) = \dfrac{t+2}{3t+4}$. Find $\phi\left(\dfrac{1}{t}\right)$.

12. Given $s = \phi(t) = \dfrac{t+2}{3t+4}$. Find $\phi(s)$ in terms of t.

26. System of coordinates. Let $X'X$ and $Y'Y$ be two straight lines meeting at right angles. Let them be considered as two number scales with the point of intersection as the zero point of each. Let P be any point in the plane. From it drop perpendiculars to the two lines. Let x represent the perpendicular to $Y'Y$, and y the perpendicular to $X'X$. If P lies to the left of $Y'Y$, x is to be considered negative. If P lies above $X'X$, then y is positive. It is clear that no matter where P is in the plane, there corresponds to it one and only one pair of perpendiculars, x and y.

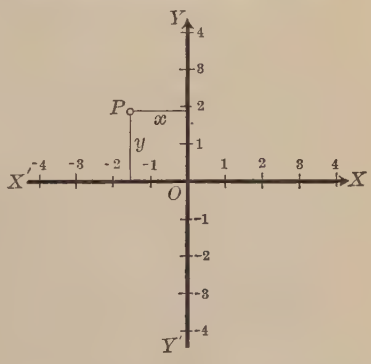

FIG. 2.

The lines of reference $X'X$ and $Y'Y$ are called the **coordinate axes**, and their intersection is called the **origin**. The first line is called the X-axis, and the second the Y-axis. The perpendicular to the X-axis from a given point in the plane is called the **ordinate** or y value of the point. The perpendicular to the Y-axis is called the **abscissa** or x value of the point.

If we have two numbers given we can find one and only one point P which has the first number for its abscissa and the second

for its ordinate. If, for example, the numbers are 2 and -5, we measure from the origin, in the positive direction, a distance 2 on the X-axis and at this point we erect a perpendicular and measure downwards a distance 5. We have then located a point whose x is 2 and whose y is -5. This point may be represented by the symbol $(2, -5)$. The symbol (a, b) denotes a point whose abscissa is a and whose ordinate is b. The symbol $P(a, b)$ is sometimes used and is read, "the point P whose coordinates are a and b."

When a point is located in the manner described above, it is said to be **plotted.** In plotting points and obtaining the geometrical pictures we are about to make, it will be convenient to use

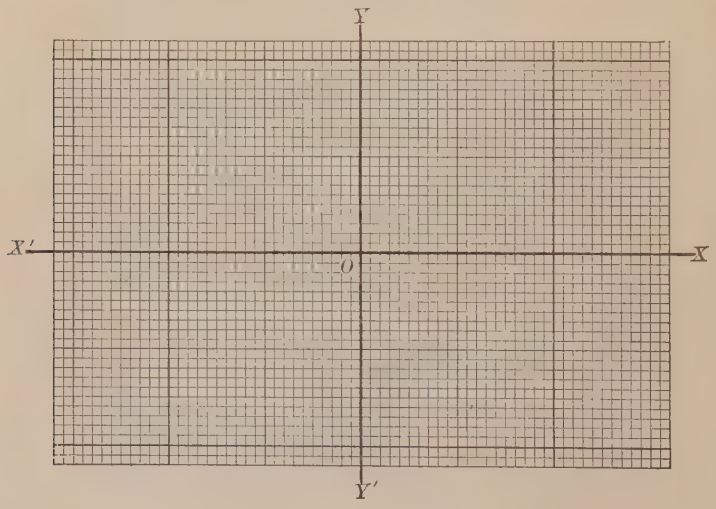

Fig. 3.

coordinate paper, which is made by ruling off the plane into equal squares with the sides parallel to the axes (Fig. 3). Then the side of a square may be taken as the unit of length to represent a number. To plot a point, count off from the origin along the X-axis the number of divisions required to represent the abscissa and from the point thus determined count off the number of divisions parallel to the Y-axis required to represent the ordinate.

EXERCISES AND PROBLEMS

1. Plot the points $(3, 4)$, $(-3, 4)$, $(-3, -4)$, $(3, -4)$, $(6, 0)$, $(-5, 0)$.

2. Draw the triangle whose vertices are $(3, -1)$, $(0, 5)$, $(-4, -2)$.

3. Draw the quadrilateral whose vertices are $(2, -2)$, $(-3, 4)$, $(-6, -3)$, $(3, 4)$.

4. If a point moves parallel to the X-axis, which of its coordinates remains constant?

5. A line joining two points is bisected at the origin. If the coordinates of one end are $(12, 3)$, what are the coordinates of the other end?

6. A square of side 3 has one corner at the point $(2, 1)$. If the sides of the square are parallel to the coordinate axes, what are the coordinates of the points that may be at the other corners of the square?

7. Given a N. and S. line and an E. and W. line for reference lines (X and Y axes respectively), the following coordinates of points of a river indicate its general course.

$(0, -1)$, $(\frac{1}{2}, -2)$, $(1, -2\frac{1}{2})$, $(2, -1\frac{1}{2})$, $(3, 1)$, $(4, 5)$, $(5, 10)$, $(-1, 0)$, $(-2, 1)$, $(-3, 2)$, $(-3\frac{1}{2}, 1)$, $(-4, -1)$, $(-5, -3)$.

Map the river from $x = -5$ to $x = +5$.

8. A square of side 4 has its center at the point $(1, 3)$. What are the coordinates of the corners (a) when the sides of the square are parallel to the coordinate axes; (b) when the diagonals are parallel to the axes? (Give answers correct to two decimal places.)

27. Graph of a function. By a method analogous to that employed in Prob. 7, Art. 26, a function may be represented with reference to coordinate axes. This representation of a function is called the **graph** of the function. The graph of $f(x)$ contains all points whose coordinates are $(x, f(x))$ and no other points.

EXAMPLE: Obtain the graph of $\frac{3}{2}x + 4$ for values of x between -5 and $+5$.

Let $f(x) = \frac{3}{2}x + 4$. The object is to present a picture which will exhibit the values of $f(x)$ which correspond to assigned values of x. Any assigned value of x with the corresponding value of $f(x)$ determines a point whose abscissa is x and whose ordinate is $f(x)$.

Assuming values for x and computing the corresponding values for $f(x)$, we obtain the following table.

x	0	1	2	$2\frac{1}{2}$	3	4	5	-1	$-1\frac{1}{2}$	-2	-3	-4	-5	$-\frac{1}{2}$	$+\frac{1}{2}$	$1\frac{1}{2}$
$f(x)$	4	$\frac{11}{2}$	7	$\frac{31}{4}$	$\frac{17}{2}$	10	$\frac{23}{2}$	$\frac{5}{2}$	$+\frac{7}{4}$	1	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	$\frac{13}{4}$	$\frac{19}{4}$	$\frac{25}{4}$

These corresponding values are plotted as coordinates of points in Fig. 4.

It should be noted that there is no limit to the number of corresponding values which we may compute and imagine plotted in a given interval along the X-axis, and further that to small changes in values of x there correspond small changes in the values of $f(x)$. These facts suggest the idea of a **continuous curve** to represent $f(x)$ much as a continuous curve is used in mapping a river. (Prob. 7, Art. 26.)

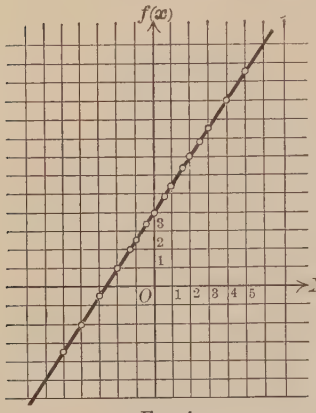

Fig. 4.

It must not, however, be assumed that all functions give continuous graphs; Article 28, below, considers a graph made up of isolated points. The important fact for this course in algebra is that we may assume a *continuous* curve for all functions which are **polynomials** in x* and for most other functions which occur in this course, although the proof of continuity is beyond the scope of this book. That is to say, it is proved in higher analysis that a function of the type

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \ (n \text{ a positive integer})$$

has a continuous graph.

Hence, in finding the graph of a polynomial, when a sufficient number of points are located to suggest the general shape of a curve through them, draw a smooth curve through the points. In particular, it is proved in analytic geometry that when $n = 1$, *the graph of a function of this type is a straight line.* In the problem in hand, the graph is the straight line shown in Fig. 4.

28. Function defined at isolated points. Much use is made of systems of coordinates in presenting statistical results when one set is a function of the other.

The following infant mortality table is made up from the

* By a polynomial in x, we mean an expression of type

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

where n is a positive integer, and a_0, a_1, $\cdots$ a_n do not contain x.

United States Life Tables of 1910. Out of 100,000 living newborn babies in each class, it shows the number of deaths during each month of the first year.

Month of First Year	White Males	White Females	Negro Males	Negro Females	City Males	Rural Males
1	4844	3787	7370	6380	4969	4570
2	1242	991	1977	1746	1370	997
3	1012	850	1831	1555	1091	822
4	863	740	1695	1394	941	699
5	750	648	1561	1252	835	595
6	673	578	1425	1134	755	515
7	610	526	1290	1036	694	459
8	553	486	1153	948	640	408
9	503	450	1037	874	586	363
10	457	421	937	800	537	325
11	420	390	857	725	496	296
12	399	359	802	663	466	277

Using the numbers of the months as abscissas and the corresponding numbers in the column headed "City Males" as ordinates, we locate the upper set of points in Fig. 5. The vertical unit is 500. The lower set of points is given by the column headed "Rural Males." In Fig. 5 we thus present to the eye the relative infant mortality of city and country children.

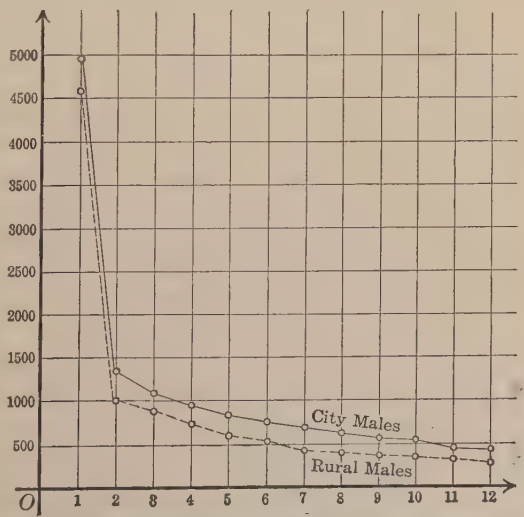

Fig. 5. — Graphical representation of infant mortality of city and rural males.

The graph in this case is made

up of 12 points. If we look upon the number of deaths as a function of the number of the month, this function is defined at only 12 points. The lines connecting the points in the figure are not necessary, but aid the eye to take in the whole situation. Where two sets of data are exhibited in the same diagram as in Fig. 5 the connecting lines prevent confusion of the two sets of points.

EXERCISES

Plot the graphs of the following functions.

1. $3x - 7$. **2.** $4x + 3$. **3.** $3x$. **4.** x.

5. $+\sqrt{25 - x^2}$.

Solution: We find the following table.

$x = 0$	1	2	3	4	4.5	5	Greater than 5
$\sqrt{25 - x^2} = 5$	4.9	4.6	4	3	2.18	0	Imaginary

$x = -1$	-2	-3	-4	-4.5	-5	Less than -5
$\sqrt{25 - x^2} = 4.9$	4.6	4	3	2.18	0	Imaginary

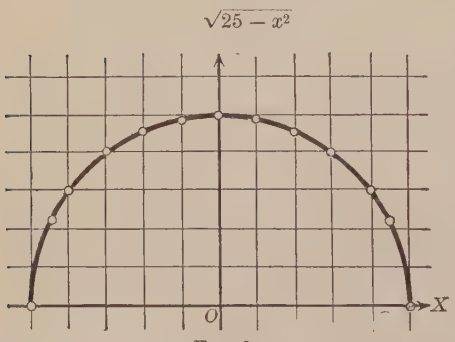

Fig. 6.

Plotting these points and drawing a smooth curve through them, we have Fig. 6.

6. $-\sqrt{25 - x^2}$.

7. $\sqrt{36 - x^2}$.

8. $\pm\sqrt{9 - x^2}$.

9. $x^2 - x - 12$.

10. $x^2 - 4$.

11. $12 + x - x^2$.

12. $2x^2 - 11x + 5$.

13. $x^2 - 2x + 1$.

14. From the table on page 37, show graphically on the same diagram the infant mortality of "white males" and of "negro males."

15. Exhibit graphically on the same diagram the infant mortality of "white males" and "white females."

16. Exhibit graphically on the same diagram the infant mortality of "negro males" and "negro females."

17. The breaking strength of ordinary manila rope is given by the formula $B = 7100\,D^2$ where B is the breaking weight in pounds and D is the diameter of the rope. Exhibit this formula graphically for the diameters $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$, 1, $1\frac{1}{8}$, $1\frac{1}{4}$, $1\frac{3}{8}$, $1\frac{1}{2}$, $1\frac{5}{8}$, $1\frac{3}{4}$, $1\frac{7}{8}$, 2 inches.

18. The following table taken from a jewelry catalogue gives the price of diamonds of the same quality for various weights. From this table give a graphical representation of the price of diamonds.

Weight in carats	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
Price in dollars	30	40	50	65	80	95	110	130	150	170

Weight in carats	0.65	0.70	0.75	0.80	0.90	1.00	1.25	1.50	1.75	2.00
Price in dollars	190	210	230	250	285	325	400	500	600	700

19. The morning and evening temperatures of a pneumonia patient were as follows: 99°, 103.2°, 105°, 103.6°, 104.2°, 105°, 104°, 105°, 103°, 104.2°, 102.3°, 97.6°, 97.4°, 98.2°, 99°, 98.2°, 98.7°, 98.4°. Give a graphical representation.

HINT: To save work in handling large numbers, 90 may be subtracted from each of the above numbers and the differences plotted. Or "degrees of fever" may be plotted — that is, degrees above 98.6°.

20. A soldier under the U. S. War Risk Insurance plan contributes towards his life insurance on a yearly term plan. This means that if he enters at age 21 he will pay in successive years while in the service the following amounts in premiums for $1000 of life insurance.

Year	1	2	3	4	5	6	7
Yearly premium in dollars	7.80	7.80	7.80	7.92	7.92	8.04	8.04

Year	8	9	10	11	12	13	14
Yearly premium in dollars	8.16	8.28	8.28	8.40	8.52	8.64	8.76

Represent graphically.

21. The postage on first-class mail matter is two cents per ounce or fraction thereof. With weights for abscissas and number of cents for ordinates exhibit this postage rate graphically.

29. Zeros of a function. By a "**zero of $f(x)$**" is meant a value of x such that the corresponding value of $f(x)$ is zero. Thus 3 and -1 are zeros of the function $x^2 - 2x - 3$, and ± 5 are zeros of $\sqrt{25 - x^2}$. Stated graphically, the "real zeros of $f(x)$" are the

abscissas of the points where the graph crosses the X-axis. In Figs. 6 and 7 the graphs and the zeros of $\sqrt{25 - x^2}$ and $x^2 - 2x - 3$ are shown. One of the main problems of algebra is the development of methods for finding the zeros of functions. The graphical solution of this problem, so far as real zeros are concerned, consists in finding where the graph crosses the X-axis. One of the advantages of the graphical method of dealing with functions is that it presents to the eye the zeros of a function.

EXERCISES

Plot and find the real zeros of the following functions.

1. $x^2 - 2x - 3$.

Solution: Compute the table.

$x =$	-3	-2	-1	0	1	2	3	4	5	···
$x^2 - 2x - 3 =$	12	5	0	-3	-4	-3	0	5	12	···

Plotting these points and drawing a smooth curve through them, we have Fig. 7. The graph crosses the X-axis at -1, and 3, which are therefore the zeros of $x^2 - 2x - 3$.

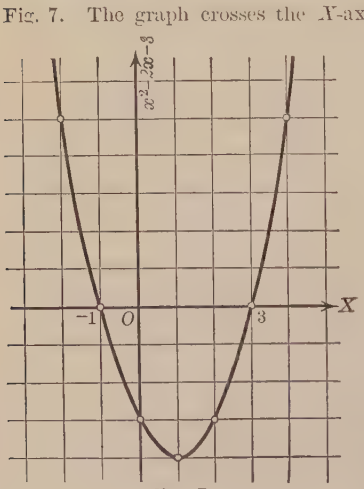

Fig. 7.

2. $3x - 5$.

3. $4x + 9$.

4. $x^2 - 6x + 5$.

5. $x^2 + 4x$.

6. $3x^2 - 11x - 4$.

7. $6x - 5 - x^2$.

8. $x^3 + 3x^2 - x - 3$.

9. $x^2 - 25x$.

10. Between what integers does each of the real zeros of $x^3 + 5x^2 - 4x - 1$ lie?

11. Show that $x^2 + x + 1$ has no real zeros.

CHAPTER IV

THE EQUATION

30. Equalities. A statement that one expression is equal to another expression is called an **equality.** The two expressions are called the **members** of the equality. There are two classes of equalities, — **identical equalities** or **identities**, and **conditional equalities** or **equations.** An identity is defined in Art. 13. It is there stated that the two members of an identity are equal for all values of the symbols for which the expressions are defined. Thus,

$$x^2 - a^2 = (x - a)(x + a), \qquad 5a = 10a - 5a$$

are identities. But in the equality $x - 5 = 4$, the two expressions $x - 5$ and 4 are equal only when x has the value 9. An equality of this kind, in which the members can be equal only for particular values of the letters involved, that is, are not equal for all values, is sometimes called an equality of condition. In this book we shall use the term equation to mean conditional equality. When it seems necessary to indicate that an equality is an identity and not a conditional equality, we shall use the sign $\equiv$ instead of the sign $=$ between the members. But the sign $=$ will be used for both identities and equations when this usage can lead to no confusion.

Which of the following equalities are identities?

(*a*) $x - 10 = 0$.

(*b*) $\dfrac{x^2 - a^2}{x + a} \equiv x - a$.

(*c*) $x^2 + 2x \equiv (x + 1)^2 - 1$.

(*d*) $x^2 + 4 \equiv (x + 2)^2 - 4x$.

(*e*) $5x^2 + 2x - 6 = 0$.

(*f*) $\dfrac{1}{1 - x} \equiv 1 + x + x^2 + x^3 + \dfrac{x^4}{1 - x}$.

In equalities (*b*) and (*f*) may x take *all* real values?

31. Definitions. In an equation there are some symbols whose values are assumed known and others whose values are unknown. These are spoken of as the **knowns** and **unknowns.** Following the conventions of elementary algebra, the first letters of the alphabet are used to represent knowns, while the last letters represent unknowns.

Any expression in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer, and a_0, a_1, a_2, $\cdots$, a_{n-1}, a_n represent any given numbers, is called a **rational integral expression in** * x, or **a polynomial in** x (cf. Art. 27). In other words, a rational integral expression in x is the algebraic sum of terms of the type kx^α, where α is restricted to take positive integral values. For example,

$$2x^2 - 5x \text{ and } \tfrac{3}{2}x^2 + 7x - \tfrac{2}{3}$$

are rational integral expressions in x.

As an extension of this definition, we define a rational integral expression in x, y, z, $\cdots$ as the algebraic sum of terms of the type

$$kx^\alpha y^\beta z^\gamma \cdots,$$

where α, β, γ, $\cdots$ are positive integers and k (called a coefficient) does not involve x, y, z, $\cdots$. For example,

$$5x^2y + 3xz + 3x^2 - 1$$

is a rational integral expression in x, y, z.

By the **degree** of a term $kx^\alpha y^\beta z^\gamma$ in any letters x, y, z, $\cdots$ is meant the sum $\alpha + \beta + \gamma + \cdots$ of the exponents of the letters in question. The **degree of a rational integral expression** is defined as that of a term whose degree is equal to or greater than that of any other term in the expression. Thus,

$$5x^2y + 3xz + 3x^2 - 1$$

is of degree two in x, one in y, one in z, one in y and z, two in x and z, three in x and y, three in x, y, and z.

* By substituting the words "function of" for "expression in" throughout this article, we obtain the definition of an important class of functions.

The statement that two rational integral expressions are equal is called a **rational integral equation.** By transposing terms, such an equation can manifestly be written in the form

$$f(x, y, z, \cdots) = 0,$$

where $f(x, y, z, \cdots)$ is a rational integral expression. The degree of $f(x, y, z, \cdots)$ in any letters is said to be the **degree of the equation** in those letters. In this course, the term degree is applied to equations only when they are in the rational integral form.

We sometimes speak of the degree of an equation without mentioning to what letters we refer. In this case, it is to be understood that we mean the degree in all the unknowns.

Equations of the first, second, third, fourth, and fifth degrees are called **linear, quadratic, cubic, quartic,** and **quintic** equations respectively.

EXERCISES

Give the degree of each of the following equations.

1. $x^2y^3 + 5x = 0.$

2. $ax^2 + bx + c = 0.$

3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

4. $y^5 = ax.$

5. Give the degree of the expression $ax^4 - 4mx^2y^2 - 3nxy + y^2$ in x. In y. In x and y.

6. Give the degree of the equation $10x^4 - 4ax^2yz - 3xyz + by^2 = 5x^4 - 2x^2y^2$ in x. In y. In z. In y and z. In x and z. In x and y. In x, y, and z.

32. Solution of an equation. To solve an equation in one unknown is to find values of the unknown that make the two members equal. Any such value is said to satisfy the equation and is called a **solution** or **root** of the equation. A solution of an equation in more than one unknown is a set of values of the unknowns which satisfy the equation. Thus, $x = 1$, $y = 2$, is a solution of $y = x + 1$.

To solve a system of equations in any number of unknowns is to find sets of values of the unknowns which will satisfy the equations. Any such set of values is said to be a solution of the system of equations.

EXERCISES

1. Is 1 a root of $x^2 - 5x + 4 = 0$?
2. Is 6 a root of $x^2 - 7x + 5 = 0$?
3. Is 4 a root of $x^2 - 2x + 4 = 0$?
4. Is $1 + i (i^2 = -1)$ a solution of $x^2 - 2x + 2 = 0$?
5. Is $x = 5$, $y = 3$, a solution of $2x + 4xy = 28$?
6. Is $x = 1$, $y = 4$, a solution of $x^2 + 3xy - y^2 = -3$?
7. Is $x = 1$, $y = 2$, $z = 1$, a solution of $3x - 2y + 5z - 2 = 0$?

33. Equivalent equations. Two equations or two systems of equations are said to be equivalent when they have the same solutions; that is, when each equation or each system is satisfied by the solutions of the other. Thus, the equations $x - 2 = 0$ and $3x - 6 = 0$ are equivalent, the second being derived from the first by multiplying both members by 3. Again, the equations $x^2 - 5x + 6 = 0$, and $-10x^2 = -50x + 60$, are equivalent. The second can be obtained from the first by performing the following operations on both numbers.

(1) Multiply both members by -10.
(2) Add $-50x + 60$ to both members.

It must not, however, be inferred when the same operation is performed on the two sides of an equation, that there necessarily results an equivalent equation. The following equations will show that this is an unwarranted inference.

1. Consider the equation $3x = x + 4$. (1)
Square both members, $9x^2 = x^2 + 8x + 16$. (2)
The equation (2) is satisfied by 2 and -1, while (1) is satisfied by 2 and not by -1. Hence, (1) and (2) are not equivalent.

2. Consider the equation $3x + 2 = 5x - 8$. (3)
Multiply both members by $(x - 1)$,
$$(x - 1)(3x + 2) = (x - 1)(5x - 8). \quad (4)$$
Equation (4) is satisfied by 1 and 5, while (3) is satisfied only by 5. Hence, (3) and (4) are not equivalent.

3. Consider the equation $\sqrt{1 - x} - x = -1$. (5)
First, add x to each member, then square both members. There results
$$1 - x = 1 - 2x + x^2. \quad (6)$$
Equation (6) is satisfied by $x = 0$ and $x = 1$; but $x = 0$ does not satisfy (5). Hence, (5) and (6) are not equivalent.

4. Consider the system of equations

$$\left.\begin{aligned} x + y &= 15, \\ x - y &= 5. \end{aligned}\right\} \tag{7}$$

Multiply the members of the first equation of (7) by x, the second by y. There results

$$\left.\begin{aligned} x(x + y) &= 15x, \\ y(x - y) &= 5y. \end{aligned}\right\} \tag{8}$$

This system (8) is satisfied by the four pairs of numbers (10, 5),* (0, 0), (0, $-$ 5), (15, 0), but (10, 5) is the only one of these pairs which will satisfy (7).

These simple examples show that the same operation performed on the two members of an equation does not necessarily lead to an equation equivalent to the original one.

It is manifestly important to know whether an equation is equivalent to that from which it is derived; and if non-equivalent, whether it contains at least all the solutions of the original equation. If a derived equation contains all the roots of the original equation and some others, we shall call it **redundant.** If the derived equation lacks some roots of the original equations, we shall call it **defective.** The student should always be on his guard against treating two equations as equivalent simply because the one has been derived from the other.

The following operations which the student has often performed in elementary algebra lead to equivalent equations:

(*a*) Adding the same number to or subtracting the same number from both members.

(*b*) Multiplying or dividing both members by the same known number provided this number is not equal to zero.

(*c*) Changing the signs of all the terms.

34. Operations that lead to redundant equations. The following operations on the two members of an equation lead, in general, to redundant equations:

(*a*) Multiplying both members by the same integral function of the unknown.

---

* The notation (10, 5) means $x = 10$, $y = 5$. (See Art. 26.)

EXAMPLE 1. Consider the equation $3x + 2 = 5x - 8$. (1)

The solution is $x = 5$.

Multiplying each member by $x - 1$, we have

$$(x - 1)(3x + 2) = (x - 1)(5x - 8). \quad (2)$$

Equation (2) has roots 5 and 1, but 1 is not a root of equation (1).

EXAMPLE 2. Consider the equation $x - 1 = 0$. (1)

The solution is $x = 1$.

Multiplying each member by x, we have

$$x^2 - x = 0. \quad (2)$$

Equation (2) has roots 0 and 1, but 0 is not a root of (1).

(*b*) Raising both members to the same integral power.

EXAMPLE 1. Take the equation $3x = x + 4$. (1)

Squaring each member, we have

$$9x^2 = x^2 + 8x + 16. \quad (2)$$

Equation (2) has roots 2 and -1, but -1 is not a root of equation (1).

EXAMPLE 2. Take the equation $-\sqrt{x} = 1$. (1)

There is no value of x that satisfies (1).

Squaring both members, we have

$$x = 1. \quad (2)$$

While 1 thus satisfies equation (2), it does not satisfy (1).

35. An operation that leads to defective equations. The following operation leads, in general, to defective equations.

Dividing both members of an equation by the same rational integral function of the unknown, when such function is a factor of each member.

EXAMPLE 1. Take the equation

$$(x^2 - 6)(x - 2) = 3x - 6. \quad (1)$$

The roots are 2, 3, and -3.

Divide both members by $x - 2$ and we have

$$x^2 - 6 = 3, \quad (2)$$

the roots of which are 3, and -3.

That is, the root 2 is lost in dividing the members by $x - 2$.

EXAMPLE 2. Take the equation $x^3 - 5x^2 + 6x = 0$.

The roots are 0, 2, and 3.

Dividing members by x, we have

$$x^2 - 5x + 6 = 0,$$

the roots of which are 2 and 3.

The root 0 is lost in dividing by x.

36. Clearing an equation of fractions. We shall call an equation **fractional** only in case some of its terms are fractions with unknowns in the denominators.

When a fractional equation is cleared of fractions, the resulting equation is generally equivalent to that from which it is derived, but it may be redundant.

Consider the equation

$$x - \frac{x^3 - 4x^2 + 5x - 2}{x^2 - 3x + 2} = x - 2. \tag{1}$$

Clearing of fractions by multiplying both members by $x^2 - 3x + 2$, we obtain $x^2 - 3x + 2 = (x - 2)(x^2 - 3x + 2)$,

which is satisfied by $x = 1$, $x = 2$, $x = 3$, while (1) is not satisfied by $x = 1$, or $x = 2$, as can be shown by substitution.

In concluding this review of equivalent equations, it need hardly be said that we have by no means exhausted all the types of operations which it may be necessary to perform on the members of an equation, but enough has been said about a few simple operations to warn the student against proceeding blindly in deriving equations from a given equation. Unless the operations on the members of an equation are known to lead to equivalent equations, the student should never regard the solution as complete until the test of substitution has been applied.

EXERCISES

1. Form equations by multiplying the members of $2x = 5$ by each of the following expressions.

(*a*) x.

(*b*) $x - 3$.

(*c*) $x^2 - 4$.

What roots have the derived equations that are not roots of $2x = 5$?

2. Form an equation by multiplying the members of $x^2 + 2 = 3x$ by $x - 3$.

In respect to what root is the resulting equation redundant?

3. What root has the equation $x^2 - x = 0$ which is not a root of the equation derived by dividing the members of the given equation by x?

4. Form an equation by dividing the members of

$$(x^2 - 6)(x - 2) = 3x - 6,$$

by $x - 2$. What root has the given equation which is not a root of the derived equation?

5. Form an equation by squaring the members of $x = 3x - 4$.

Show that 1 is a root of the derived equation but not of the given equation.

6. Given an equation whose members are rational integral functions of x. If you multiply the members by $x - a$ where a is not a root of the given equation, what root is introduced into the derived equation? Illustrate with the given equation $x = b$.

7. If $x - a$ is a factor of each member of a given equation, what root of the given equation is, in general, lacking in the equation obtained by dividing the members of the given equation by $x - a$? Illustrate with the given equation $x(x - a) = b(x - a)$.

Reduce the following equations to rational integral equations and discuss the question of equivalence.

8. $\dfrac{5 - 3x}{x + 1} = \dfrac{7 - 9x}{1 + 3x}$.

9. $\dfrac{3x - 5}{4x - 7} = \dfrac{3x - 7}{4x - 5}$.

10. $\dfrac{3}{x^2 - 25} + \dfrac{1}{x + 5} = \dfrac{2}{5 - x}$.

11. $\sqrt{x - 1} - 1 = \sqrt{x - 4}$.

12. $\sqrt{x} - 1 = -\sqrt{x - 9}$.

13. $\sqrt{x - 15} = 1 + \sqrt{x}$.

14. $1 - \dfrac{x + 5}{2x + 1} = \dfrac{x - 6}{x - 2}$.

15. $\dfrac{x}{x^2 - 1} = \dfrac{15 - 7x}{8(1 - x)}$.

16. $x = 21 + \sqrt{x^2 - 9}$.

17. $\dfrac{x^2 - 5x + 4}{x^2 - x} = 1$.

CHAPTER V

LINEAR EQUATIONS

37. Type form. Any linear equation (Art. 31) in one unknown

$$ax + b = 0, \quad a \neq 0, \tag{1}$$

can be put into the form $x = -\frac{b}{a}$.

The point which represents $-\frac{b}{a}$ on the line of Fig. 1 may be conveniently regarded as the *locus* of equation (1) in one-dimensional space.

A linear equation in two unknowns

$$ax + by + c = 0, \quad b \neq 0, \tag{2}$$

can be put into the form

$$y = -\frac{ax}{b} - \frac{c}{b}. \tag{3}$$

Since in (3) we may assign to x any value and compute a corresponding value for y, the equation defines y as a function of x in accordance with our definition of a mathematical function (Art. 24).

The graph of a linear function has been discussed in Art. 27, and this graph may be conveniently regarded as the *locus* of equation (2) in two-dimensional space.

The locus in two-dimensional space of an equation in two variables consists of all points whose coordinates satisfy the equation and of such points only.

EXERCISES

Plot the loci of the following equations.

1. $x - y = 1$.

SOLUTION: Putting this in form (3), we obtain

$$y = x - 1.$$

The graph of the function $x - 1$ is shown in Fig. 8, and by definition this is the locus of the equation. Since the graph of any rational integral function of degree one (Art. 27) is a straight line, the locus of any linear equation is a straight line.

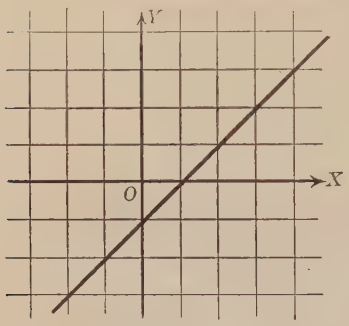

Fig. 8.

2. $2x - y = 1$.

3. $x - 3y = 1$.

4. $3x + 4y = 4$.

5. $5x + 3y = 0$.

6. $7x - 5y = 0$.

7. $7x + 5y = 4$.

8. $\dfrac{x+y}{2} + \dfrac{x-y}{3} = 8$.

9. $4x - 3y = 36$.

38. Simultaneous linear equations.

Let
$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

be two linear equations in two unknowns. Multiply the members of the first by b_2 and those of the second by $-b_1$. Adding the members of the two resulting equations, we obtain

$$(a_1b_2 - a_2b_1)x = (b_2c_1 - b_1c_2),$$

or
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

In a similar manner, by multiplying the first and second equations by $-a_2$ and a_1 respectively, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

We note that the denominators of the above fractions are alike. This denominator may be denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is called a **determinant.** Since it has two rows and two columns, it is said to be of the second order. The letters a_1, b_1, a_2, b_2, are called the **elements** of the determinant, and a_1, b_2, constitute the **principal diagonal.** A determinant of the second order then represents the number which is obtained by subtracting

from the product of the terms in the principal diagonal, the product of the other two terms. Thus,

$$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = x\,w - y\,z, \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

Using the determinant notation, we may now write the solutions of our equations in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

We note that the numerator of the solution for x is obtained from the denominator by substituting in place of a_1, a_2, which are the coefficients of x in the equations to be solved, the known terms c_1, c_2. In a similar manner, in the numerator of the solution for y we replace b_1, b_2 by c_1, c_2 respectively.

EXERCISES

1. Solve: $x + y = 3$,
$2x + 3y = 1$.

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{9 - 1}{3 - 2} = 8,$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 - 6}{3 - 2} = -5.$$

Solve the following pairs of equations, using determinants.

2. $3x + 2y = 23$,
$5x - 2y = 29$.

3. $\frac{x}{2} + \frac{y}{5} = 11$,
$\frac{x}{4} + \frac{y}{5} = 7$.

4. $6x + 5y = 16$,
$5x + 12y = -19$.

5. $3x - 6y = 9$,
$2x + 7y = 28$.

6. $\frac{11x - 5y}{22} = \frac{3x + y}{32}$,
$8x - 3y = 23$.

7. $\frac{x + y}{3} + \frac{y - x}{2} = 9$,
$\frac{x}{2} + \frac{x + y}{9} = 5$.

39. Graphical solution of a system of linear equations. As stated in Art. 37, the locus of any linear equation in x and y is a straight line. Any such equation is satisfied by an indefinitely large number of pairs of values of x and y. That is, by the coordinates of all points on its locus. In the graphical solution of the system of two equations, we seek the coordinates of points common to the loci of the two equations.

As the loci are two straight lines, three cases arise:

(1) In general, two lines intersect in one and only one point.

(2) Two lines may be parallel, and thus have no point in common.

(3) Two lines may be coincident, and thus have an indefinitely large number of points in common.

Corresponding to these three cases, a system of two linear equations has, in general, one and only one solution, but it may have no solution or an indefinitely large number of solutions. When the loci are two parallel lines, there is no pair of numbers which satisfies both equations, and the equations are said to be *incompatible* or *inconsistent.*

Referring to the expressions for x and y (Art. 38) in determinant form, we see that there is one common solution of the equations unless the determinant in the denominator is zero. When the loci are two coincident lines, the two equations of the system are equivalent.

EXERCISES

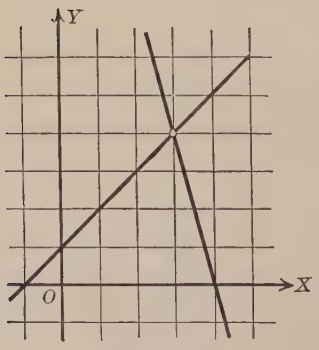

Fig. 9.

Find the solutions of the following equations by determinants and by plotting the loci.

1. $x - y + 1 = 0,$
$4x + y - 16 = 0.$

See Fig. 9.

2. $x + y = 2,$
$3x - 2y = 1.$

3. $2x + 3y = 18,$
$3x - 2y = 1.$

4. $x + 3y = 10,$
$3x + 2y = 9.$

5. $2x - 9y = 11,$
$3x + 12y = 15.$

6. $2x - y = 1,$
$x + 3y = 11.$

7. $3x - 12y = -12,$
$x + 8y = 20.$

8. $6x + 5y = 16,$
$5x - 12y = -19.$

9. $\frac{x}{3} + \frac{y}{5} = 8,$
$\frac{x}{9} - \frac{y}{10} = 1.$

10. $2x + 2y = 4,$
$3x + 3y = 6.$

11. $x + y = 2,$
$5x + 5y = 20.$

12. $9x - 4y = 0,$
$3x + 8y = 7.$

13. $5x + 7y = 3,$
$3x + 14y = 6.$

14. $7x - y = 33,$
$12y - x = 19.$

15. In solving a system of equations

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$$

by determinants, show that if the determinant in the denominator is zero, and the determinant in one numerator is likewise zero, then the two equations are equivalent.

40. Determinants of the third order. The square array of nine numbers with bars on the sides

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is a convenient abbreviation for the expression

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3, \qquad (1)$$

and is called a determinant of the third order. As in the case of the determinant of the second order, the letters a_1, b_1, etc., are called the elements, and the letters a_1, b_2, c_3 form the principal diagonal. The expression (1) is called the expansion or development of the determinant. It is seen that each term of the expansion consists of the product of three elements, no two of which lie in the same row or in the same column. Any determinant of the third order

may be easily expanded as follows. Rewrite the first and second columns to the right of the determinant. The diagonals

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

running down from left to right give the positive terms. The diagonals running down from right to left give the negative terms.

EXERCISES

Obtain the expansions of the following determinants.

1. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1\cdot 7\cdot 5 + 3\cdot 3\cdot 1 + 4\cdot 3\cdot 2 - 4\cdot 7\cdot 1 - 3\cdot 3\cdot 1 - 5\cdot 2\cdot 3 = 1.$

2. $\begin{vmatrix} 10 & 2 & 8 \\ 5 & 4 & 0 \\ 3 & 1 & 7 \end{vmatrix}$. **3.** $\begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & -4 \\ 8 & 0 & 5 \end{vmatrix}$. **4.** $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 1 & 4 \\ 6 & 2 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 2 & 4 \\ -2 & -1 & -4 \\ 3 & -3 & 5 \end{vmatrix}$. **6.** $\begin{vmatrix} a & b & 0 \\ 0 & x & y \\ u & 0 & v \end{vmatrix}$. **7.** $\begin{vmatrix} x & 1 & 3 \\ 2 & x & 4 \\ 6 & 3 & 1 \end{vmatrix}$. **8.** $\begin{vmatrix} x & 2 & x \\ 1 & 1 & 1 \\ x & 4 & x \end{vmatrix}$.

41. Solution of three equations with three unknowns. Let the three equations be

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$

$$a_2x + b_2y + c_2z = d_2, \tag{2}$$

$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

Multiplying (1) and (2) by b_2 and $-b_1$ respectively and adding, we get

$$(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = d_1b_2 - d_2b_1. \tag{4}$$

Eliminating y in a similar manner from (1) and (3), we find

$$(a_3b_1 - a_1b_3)x + (c_3b_1 - b_3c_1)z = d_3b_1 - d_1b_3. \tag{5}$$

We now have two equations in two unknowns, x and z. Eliminating z from these two, we find

$$\begin{aligned}[(a_1b_2 - a_2b_1)(c_3b_1 - b_3c_1) - (a_3b_1 - a_1b_3)(b_2c_1 - b_1c_2)]x \\ = (d_1b_2 - d_2b_1)(c_3b_1 - b_3c_1) - (d_3b_1 - d_1b_3)(b_2c_1 - b_1c_2),\end{aligned}$$

which after some simplification gives us

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_3b_2c_1 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3}.$$

The denominator is the development of the determinant in Art. 40, while the numerator is the same as the denominator with a replaced by d. Hence we can write the solution for x in the form

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

provided the determinant in the denominator is not zero.

In a similar way, we can find the values of y and z.

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The denominators in the expressions for x, y, and z are the same, while the numerators are obtained from the denominators by replacing the coefficients of the unknown in question by the known terms. For example, in the numerator of y, the knowns d_1, d_2, d_3 replace b_1, b_2, b_3 respectively.

EXERCISES

Solve:

1.

$$x - y - z = -6,$$
$$2x + y + z = 0,$$
$$3x - 5y + 8z = 13.$$

SOLUTION:

$$x = \frac{\begin{vmatrix} -6 & -1 & -1 \\ 0 & 1 & 1 \\ 13 & -5 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{-78}{39} = -2,$$

$$y=\frac{\begin{vmatrix}1&-6&-1\\2&0&1\\3&13&8\end{vmatrix}}{\begin{vmatrix}1&-1&-1\\2&1&1\\3&-5&8\end{vmatrix}}=\frac{39}{39}=1, \qquad z=\frac{\begin{vmatrix}1&-1&-6\\2&1&0\\3&-5&13\end{vmatrix}}{\begin{vmatrix}1&-1&-1\\2&1&1\\3&-5&8\end{vmatrix}}=\frac{117}{39}=3.$$

2. $x+y+z=6,$
$3x-y+2z=7,$
$4x+3y-z=7.$

3. $3x+4y-5z=32,$
$4x-5y+3z=18,$
$5x-3y-4z=2.$

4. $5x-4y+2z=48,$
$3x+3y-4z=24,$
$2x-5y+3z=19.$

5. $2x+3y+z=4,$
$x+2y+2z=6,$
$5x+y+4z=21.$

6. $\frac{1}{x}+\frac{1}{y}=1,$
$\frac{1}{x}+\frac{1}{z}=1,$
$\frac{1}{y}+\frac{1}{z}=1.$

7. $2x+5y-3z=17,$
$6x-2y-5z=-3,$
$3x+7y+4z=-18.$

8. $x+2y-z=1,$
$3x+2y-4z=7,$
$x+2y-z=3.$

9. $x+\frac{1}{2}y+\frac{1}{3}z=10,$
$\frac{1}{2}(x+z)+y=9,$
$\frac{1}{4}(x-z)-2y+7=0.$

MISCELLANEOUS EXERCISES AND PROBLEMS

1. Show that $\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}=a_1\begin{vmatrix}b_2&c_2\\b_3&c_3\end{vmatrix}+b_1\begin{vmatrix}c_2&a_2\\c_3&a_3\end{vmatrix}+c_1\begin{vmatrix}a_2&b_2\\a_3&b_3\end{vmatrix}.$

2. Show that $\begin{vmatrix}a_1+a_2&c_1\\b_1+b_2&c_2\end{vmatrix}=\begin{vmatrix}a_1&c_1\\b_1&c_2\end{vmatrix}+\begin{vmatrix}a_2&c_1\\b_2&c_2\end{vmatrix}.$

3. Develop $\begin{vmatrix}a_1&b_1&c_1\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}.$

4. Develop $\begin{vmatrix}x&2&y\\1&1&1\\x&4&y\end{vmatrix}.$

5. Solve for x $\begin{vmatrix}x&8\\3&4\end{vmatrix}=0.$

6. Solve for x $\begin{vmatrix}x&4&1\\x&3&2\\2&1&5\end{vmatrix}=0.$

7. Solve for x and y the system of equations

$$\begin{vmatrix}x&1&y\\1&-1&1\\-1&2&0\end{vmatrix}=0, \qquad \begin{vmatrix}x&1&1\\y&-1&0\\-3&2&1\end{vmatrix}=0.$$

8. A certain kind of wine contains 25 per cent of alcohol and another kind contains 30 per cent. How many gallons of each must be mixed to make 50 gallons of the mixture 27 per cent alcohol?

9. What amounts of silver 72 per cent pure and 84.8 per cent pure must be mixed to get 8 ounces of silver 80 per cent pure?

10. The sum of the three angles of a triangle is 180°; find the two acute angles of a right-angled triangle if one of them is four times the other.

11. If the sides of a rectangular field were each increased 10 feet, the area would be increased 14,500 square feet. If the length were increased 10 feet and the width decreased 10 feet, the area would be diminished 1700 square feet. Find the area of the field.

12. Two numbers are written with the same two digits; the difference of the two numbers is 45, and the sum of the digits is 9. What are the numbers?

13. A six-figure number has 1 for the last figure. If this last figure is removed and placed before the others, a new six-figure number is made whose value is one third the original number. What is the original number?

14. The planet Mercury makes a circuit about the sun in 3 months. Venus makes the circuit in $7\frac{1}{2}$ months. Find the number of months between two successive times when Mercury is between Venus and the sun.

15. In Wilson and Gray's determination of the temperature of the sun the Fahrenheit reading of the temperature is 5552 more than the centigrade reading. What is the centigrade reading?

16. If h represent the height in meters above sea level, and b represent the reading of a barometer in millimeters, it is known that $b = k + hm$, where k and m are constants. At a height 120 meters above sea level the barometer reads 751, at height 769 meters it reads 695. What is the formula showing the relation between b and h?

17. Two runners are practicing on a circular track 126 yards in circumference. When running in opposite directions they meet every 13 seconds. Running in the same direction, the faster passes the slower every 126 seconds. How many minutes does it take each to run a mile?

18. The relation between the boiling point w of water in degrees Fahrenheit and h the height in feet above sea level is known to be of the form $x - wy = h$, where x and y are numbers to be determined by experiment. It is observed at the height 2200 feet that the boiling point is 208° F. At sea level the boiling point is 212° F. What is the formula showing the relation between w and h?

19. It is required to find the amount of expansion of a brass rod for a rise in temperature of one degree centigrade, also the length of the rod at a temperature 0°. If c represent the expansion, and b_0 the length required, it is known that $b = ct + b_0$, where b is the length of the bar at the temperature t. When $t = 20°$, the length of the rod is 1000.22; when $t = 60°$, the length is 1001.65.

20. A man has $35,000 at interest. For one part he receives $3\frac{1}{2}$ %, for the other 4 %. His income from this money is $1300 per year. How is the money divided?

21. To find the average grade of a freshman in mathematics, his grade in analytic geometry is multiplied by 5, his grade in algebra by 3, and his grade in trigonometry by 2, and the sum of the three products is divided by 10. This gives 89 for the average grade. If the grades in analytic geometry and algebra had been interchanged, his average grade would have been 91. If the three studies had all counted the same number of credits, his grade would have been 90. What are the grades in each of the three studies?

22. A cistern is filled with three pipes. The first and second will fill it in 72 minutes, the second and third in 120 minutes, and the first and third in 90 minutes. How long will it take each of the pipes to fill it?

23. Four numbers have the property, that when successively the average of three is added to the fourth, the numbers 29, 23, 21, and 17 result. What are the numbers?

24. Five numbers are arranged in order of magnitude. The difference between any two consecutive numbers is the same number. The sum of the numbers is 60. What positive integers satisfy these conditions?

25. Some books are divided among 3 boys, so that the first had 12 less than half of all the books, the second had one less than half the remainder, and the third had 17. Find the number each received.

26. How many 5 per cent bonds of $100 each at 90 must I sell in order that by investing the proceeds in 6 per cent stock at 102 my income may be increased $300?

27. Between two towns the road is level for half the distance. The speeds on a bicycle are 3, 6, and 9 miles an hour uphill, on the level, and downhill, respectively. It takes $5\frac{3}{4}$ hours to go and $4\frac{3}{4}$ hours to return. What are the lengths of the level and inclined parts of the road?

CHAPTER VI

QUADRATIC EQUATIONS

42. Type form. Any equation of the second degree (Art. 31) in one unknown x can, by transforming and collecting terms, be written in the typical form

$$ax^2 + bx + c = 0,$$

where a, b, c do not involve x, and have any values with the one exception that a is not zero. Since the result of multiplying the members of an equation in this typical form by any given number is an equation in typical form, the a, b, c can be selected in an indefinitely large number of ways. In particular, since we can change the signs of all the terms in an equation, we may assume that a is positive when it is a real number.

The function $ax^2 + bx + c$ $(a \neq 0)$ is called the typical quadratic function.

EXERCISES

Arrange the following equations in the typical form and select a, b, and c from the resulting equations.

1. $4x^2 - 5 + \frac{3x}{2} = \frac{2x^2}{3} + m.$

By transposing and collecting terms,

$$\frac{10}{3}x^2 + \frac{3x}{2} - (5 + m) = 0,$$

so that $$a = \frac{10}{3},\ b = \frac{3}{2},\ c = -(5 + m).$$

2. $x^2 + (2x + 5)^2 = 3x.$

3. $\frac{x^2}{9} + \frac{(x-1)^2}{16} = 1.$

4. $x^2 - 2dx - \frac{1}{2}x + d^2 + \frac{d}{2} - \frac{1}{2} = 0.$

5. $4m^2x^2 + 3k^2x^2 - 8mx + 3x - m + k = 0.$

6. $(y + 3)^2 + (y - 2)^2 = 0.$

7. $r^2 + (2r - 7) = 10.$

8. $(z + 2)^3 - (z - 3)^3 - 1 = 0.$

9. $u^2 + (mu + n)^2 = k.$

43. Solution of the quadratic equation. A quadratic equation may be solved by the process of "completing the square."

For example, to solve $3x^2 + 5x - 2 = 0$,

write the equation in the form $x^2 + \frac{5}{3}x = \frac{2}{3}$.

Add $(\frac{1}{2} \cdot \frac{5}{3})^2 = \frac{25}{36}$ to both members, and the left-hand member is a perfect square. That is,

$$x^2 + \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{2}{3} + \tfrac{25}{36} = \tfrac{49}{36},$$

or

$$(x + \tfrac{5}{6})^2 = \tfrac{49}{36}.$$

Extract the square root, $x + \frac{5}{6} = \pm \frac{7}{6}$,

$$x = -2, \text{ or } \tfrac{1}{3}.$$

Both of these values of x satisfy the original equation, as we see on substituting them for x. Thus

$$3(-2)^2 + 5(-2) - 2 = 3 \cdot 4 - 10 - 2 = 0.$$

$$3(\tfrac{1}{3})^2 + 5(\tfrac{1}{3}) - 2 = 3 \cdot \tfrac{1}{9} + \tfrac{5}{3} - 2 = 0.$$

Apply this method to the general quadratic equation

$$ax^2 + bx + c = 0.$$

Transpose c and divide by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ to both members to make the left-hand member a perfect square,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extract the square root,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots of the general quadratic equation

$$ax^2 + bx + c = 0$$

are $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

as may be verified by substitution. The expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

may therefore be used as a formula for the solution of any quadratic equation. Thus, to solve the equation

$$3x^2 + 5x - 2 = 0,$$

we substitute in the formula, $a = 3$, $b = 5$, $c = -2$ and find

$$x_1 = \frac{-5 + \sqrt{25 - 4 \cdot 3 \cdot (-2)}}{6} = \frac{-5 + \sqrt{49}}{6} = \frac{1}{3}.$$

Similarly

$$x_2 = \frac{-5 - \sqrt{49}}{6} = -2.$$

EXERCISES

Solve the following equations by use of the formula, and verify by substitution.

1. $2x^2 + x - 1 = 0$.
2. $2x^2 - 3x - 2 = 0$.
3. $4x^2 - 9x + 2 = 0$.
4. $3x^2 + x - 2 = 0$.
5. $2x^2 + 3x - 9 = 0$.
6. $x^2 - 6x - 7 = 0$.
7. $3x^2 + 8x - 3 = 0$.
8. $x^2 + 3x + 5 = 0$.
9. $2x^2 - 5x + 2 = 0$.
10. $2x^2 + 7 = 4x$.
11. $7x^2 + 7x + \frac{7}{4} = 0$.
12. $s^2 + 12 = 8s$.
13. $3s^2 + 2 = 6s$.
14. $3t^2 + 2t = 4$.
15. $7y^2 + 9y - 10 = 0$.
16. $3r^2 + r = 200$.
17. $6x^2 + 5x = -1$.
18. $9x^2 + 3x = 2$.
19. $7x^2 + 2x = 32$.
20. $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}$.
21. $8x + 11 + \frac{7}{x} = \frac{68x}{7}$.
22. $\frac{x + \frac{1}{x}}{x - \frac{1}{x}} + \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{13}{4}$.

23. $x^2 + x\sqrt{5} - \sqrt{5} = 0.$

24. $\begin{vmatrix} x & 1 & 8 \\ 2 & x & 3 \\ 1 & 0 & 3 \end{vmatrix} = 0.$

25. $\begin{vmatrix} 1 & x & 2 \\ 1 & 1 & 4 \\ x & 1 & -1 \end{vmatrix} = 1.$

26. $x^2 + ax - 2a^2 = 0.$

27. $x^2 - 2ax + 3x - 6a = 0.$

28. $2x^2 + 3nx - 2n^2 = 0.$

29. $x^2 + 2ax + bx + 2ab = 0.$

30. Show by substitution that $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are roots of $ax^2 + bx + c = 0$.

44. Solution by factoring. When the left-hand member of a quadratic equation can be factored readily, this is the easiest method of solution. Take, for example, the equation

$$x^2 - 4x - 21 = 0.$$

The factors of the left-hand member are easily found. They are $(x + 3)$ and $(x - 7)$, and we may write our equation in the form

$$(x + 3)(x - 7) = 0.$$

Any value of x which makes either factor zero will satisfy the equation. If $x = -3$, we have

$$(-3 + 3)(-3 - 7) = 0 \cdot (-10) = 0.$$

Again if $x = 7$, we have

$$(7 + 3)(7 - 7) = 10 \cdot 0 = 0.$$

Hence, -3 and 7 are solutions of the given quadratic equation.

EXERCISES

Solve the following by factoring.

1. $x^2 - 9x + 14 = 0.$

2. $x^2 + 8x + 7 = 0.$

3. $x^2 - x - 20 = 0.$

4. $4x^2 + 4x + 1 = 0.$

5. $20x^2 + 11x - 3 = 0.$

6. $3x^2 - 13x = 10.$

7. $2s^2 - s - 36 = 0.$

8. $2a^2 + a - 3 = 0.$

9. $7t^2 + 10t - 8 = 0.$

10. $6y^2 + 35y - 6 = 0.$

Solve by any method.

11. $3x^2 - 15x = 42.$

12. $x^2 + 4x + \frac{25}{4} = 0.$

13. $x^2 - 6x + 10 = 0$.

14. $4x^2 - 28x + 49 = 0$.

15. $9x^2 - 27x - 70 = 0$.

16. $x^2 + x - a^2 - a = 0$.

17. $x^2 + (5 - x)^2 = (5 - 2x)^2$.

18. $x^2 - 6x + 4 = 0$.

19. $3x^2 + 2x + 1 = 0$.

20. $(1 - e^2)x^2 - 2mx + m^2 = 0$.

21. $\sqrt{x + 16} = x - 4$.*

22. $x + \sqrt{x + 6} = 14$.

23. $\sqrt{10x - 34} + 2\sqrt{x + 4} = \sqrt{2(3x + 35)}$.

24. $\sqrt{27 - x} = \sqrt{x + 2} + \sqrt{3x + 3}$.

25. $\sqrt{4x - 5 + \sqrt{2x - 9}} = 4$.

26. $\sqrt{x + 11 + \sqrt{\left(\frac{x}{5}\right)^2 + 5x + 50}} = 9$.

27. $\sqrt{\frac{x - 7}{x - 19}} = \sqrt{x - 19}$.

28. $\sqrt{18x + 5} - 2\sqrt{3x} = \sqrt{2}$.

29. $\sqrt{2}\sqrt{x - 1} + \sqrt{34 - x} = 9$.

30. $\sqrt{x} - \sqrt{2x + 1} = 1$.

Solve the following equations to two significant figures.

31. $x^2 - 1.83x + 0.81 = 0$.

32. $x^2 - 0.91x - 6.66 = 0$.

33. $0.001x^2 - 0.01x - 0.1 = 0$.

34. $2.1x^2 + 10.3x - 5.8 = 0$.

45. Equations in the quadratic form. If in an equation we may replace an expression containing the unknown by a new letter and have a quadratic equation in that letter, then the original equation is said to be in the **quadratic form.** Thus, in the equation

$$x - 3 - \sqrt{x - 3} - 2 = 0,$$

if we let $z = \sqrt{x - 3}$, we have $z^2 - z - 2 = 0$.

In $$2x^{-3} + x^{-\frac{3}{2}} + 1 = 0,$$

if we let $z = x^{-\frac{3}{2}}$, we have $2z^2 + z + 1 = 0$.

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* HINT: Square both sides. In solving such problems the results should be tested in every case, remembering that the radical stands for the positive square root of the number under it. Why should the results be tested?

EXERCISES

Solve the following equations and check results.

1. $\sqrt{x+10}+\sqrt[4]{x+10}=2.$

Solution: Let $z=\sqrt[4]{x+10}.$

The equation then becomes

$$z^2+z-2=0,$$

or $$z=1, \text{ or } -2.$$

Replacing z by its value in terms of x, we have

$$\sqrt[4]{x+10}=1,$$

or $$\sqrt[4]{x+10}=-2,$$

$$x+10=1,\ x=-9.$$

$$x+10=16,\ x=6.$$

Check: $$\sqrt{-9+10}+\sqrt[4]{-9+10}=2,$$

$$1+1=2.$$

$$\sqrt{6+10}+\sqrt[4]{6+10}=4+2,$$

$$\neq 2.$$

Hence, the result $x=-9$ satisfies the equation to be solved, but the result $x=6$ does not satisfy it.

2. $x^4-13x^2+36=0.$

3. $(4x^2-3)^2+(8x^2-6)^2=80.$

4. $\dfrac{x^2+2}{x^2-2}+\dfrac{x^2-2}{x^2+2}=\dfrac{10}{3}.$

5. $(x^2+5x)^2+x(x+5)=42.$

6. $\left(x+\dfrac{1}{x}\right)^2+4x+\dfrac{4}{x}=12.$

7. $\sqrt{x+16}=4.$

8. $x+\sqrt{x+6}=14.$

9. $\sqrt{2x+1}-x=\frac{1}{2}.$

10. $\sqrt{x}=8-\dfrac{15}{\sqrt{x}}.$

11. $\sqrt{\dfrac{7-2x}{7+2x}}+\sqrt{\dfrac{7+2x}{7-2x}}=\dfrac{3}{2}\sqrt{2}.$

12. $2x^2-4x+3\sqrt{x^2-2x+6}=15.$

13. $x^6+7a^3x^3-8a^6=0.$

14. $x^4+2x^3-x^2-2x-3=0.$*

15. $x^4-8x^3+23x^2-28x-8=0.$

16. $x^3+7x^{\frac{3}{2}}-8=0.$

17. $x^{-3}-9x^{-\frac{3}{2}}+8=0.$

18. $x^{-5}+31x^{-\frac{5}{2}}-32=0.$

19. $3\sqrt[3]{x^2}+6\sqrt[3]{x}-4=0.$

20. $x^3-8=0.$

21. $ax^{2n}+bx^n+c=0.$

22. $2-6x^{-2}-x^{-1}=0.$

23. $x^{\frac{3}{2}}-4x-21\sqrt{x}=0.$

---

* Hint: Write the equation in the form

$$(x^4+2x^3+x^2)-2(x^2+x)-3=0.$$

24. $\dfrac{x}{x^2+1}+\dfrac{x^2+1}{x}=\dfrac{5}{2}$.

25. $(x+1)(x+2)(x+3)(x+4)=24$.*

46. Theorems concerning the roots of quadratic equations.

THEOREM I. *If r is a root of the equation*

$$ax^2+bx+c=0, \tag{1}$$

then $x-r$ is a factor of the left-hand member. Conversely, if $x-r$ is a factor of the left-hand member, then r is a root of the equation.

If r is a root of the equation,

$$ar^2+br+c=0, \tag{2}$$

then

$$ax^2+bx+c=ax^2+bx+c-(ar^2+br+c) \tag{3}$$

$$=a(x^2-r^2)+b(x-r) \tag{4}$$

$$=(x-r)(ax+ar+b). \tag{5}$$

Hence, $x-r$ is a factor of ax^2+bx+c.

If $x-r$ is a factor of ax^2+bx+c, then the substitution of r for x makes the factor $x-r$ vanish, and r is a root of

$$ax^2+bx+c=0.$$

EXERCISES

Form quadratic equations of which the following are roots.

1. 3, 1.

SOLUTION: When the right-hand member of the equation to be formed is 0, the left-hand member has factors $x-3$ and $x-1$. Hence,

$$(x-3)(x-1)=x^2-4x+3=0$$

is a quadratic equation with roots 1 and 3.

2. 3, -2.

3. 7, 0.

4. $\sqrt{5}$, -5.

5. $\sqrt{5}$, $-\sqrt{5}$.

6. $\sqrt{5}-1$, $\sqrt{5}+1$.

7. i,† $-i$.

---

* HINT: Multiply first and fourth, and second and third factors together and write in the form

$$[(x^2+5x)+4][(x^2+5x)+6]=24.$$

† In these exercises, $i^2=-1$. (See Art. 22.)

8. $4+3i, 4-3i.$ **9.** $2+5\sqrt{3}, 2-5\sqrt{3}.$

10. $-\frac{1}{2}+\frac{i\sqrt{3}}{2}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}.$ **11.** $2, \frac{1}{2}.$

12. $\frac{1}{a}, a.$ **13.** $\frac{m}{n}, \frac{n}{m}.$

14. Verify by performing the indicated operations that

$$a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=ax^2+bx+c.$$

47. Number of roots. In order to avoid certain exceptions, an equation $f(x)=0$ is said to have as many roots as $f(x)$ has factors of the type $x-r_1$ where r_1 is any number. A factor $x-r_1$ may be repeated. For example, if $(x-r_1)^2$ is a factor of $f(x)$, we say that $f(x)=0$ has two roots equal to r_1.

We have shown that a quadratic equation has two roots. The question arises: has it *only* two or may it have more? This question is answered by the following

THEOREM II. *A quadratic equation has only two roots.*

Proof. Suppose there is, in addition to

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

a third root r_3, distinct from r_1 and r_2, of the equation

$$ax^2+bx+c=0.$$

By Ex. 14, Art. 46, $\quad ax^2+bx+c=a(x-r_1)(x-r_2).$

Hence if r_3 is a root,

$$a(r_3-r_1)(r_3-r_2)=0.$$

But this is impossible since no one of these factors is zero.

(III, Art. 5.)

48. Special or incomplete quadratics. If b or c is zero in the quadratic equation $ax^2+bx+c=0$, the equation is said to be incomplete.

I. When $c = 0$, $ax^2 + bx = 0$ is the typical form of the equation. We can always write this equation in the form

$$x(ax + b) = 0.$$

Hence, the roots are 0 and $-\frac{b}{a}$. Conversely, if 0 is a root of a quadratic equation $ax^2 + bx + c = 0$,

then $$a \cdot 0 + b \cdot 0 + c = 0.$$

That is, $$c = 0.$$

Therefore, *a quadratic equation has a root equal to zero when and only when the equation has no known term.*

II. When $b = 0$, $ax^2 + c = 0$ is the typical form.

In this case, $$x = \pm\sqrt{-\frac{c}{a}}.$$

Conversely, if the roots of a quadratic equation are arithmetically equal, but opposite in sign, there is no term containing x in the first degree; for, if $+r$ and $-r$ are both roots of

$$ax^2 + bx + c = 0,$$

we have $$ar^2 + br + c = 0,$$

$$ar^2 - br + c = 0,$$

$$2br = 0.$$

Since $$r \neq 0,$$

it follows that $$b = 0.$$

Hence, *a quadratic equation has two roots arithmetically equal but opposite in sign when and only when the term in x vanishes.*

III. When $b = 0$, $c = 0$, the typical form is $ax^2 = 0$. *Both roots of a quadratic equation are equal to zero when and only when the known term and the term in x vanish.*

EXERCISES

Determine k so that each of the following equations shall have one root equal to zero.

1. $5x^2 - 16x + 5 - k^2 = 0.$
2. $10x^2 + 14x + 2k - 16 = 0.$
3. $10y^2 - 16y + k^2 - 4k + 3 = 0.$

Determine k and m so that each of the following equations shall have two roots equal to zero.

4. $5x^2 - 16mx + kx - 4m + k + 6 = 0.$
5. $3z^2 - 8mz + 4kz + 6z + 4m + 2k + 1 = 0.$
6. $x^2 + (2m + 1)x + 4k^2 + 2k = 0.$

Determine k so that the roots may be arithmetically equal, but opposite in sign.

7. $x^2 + 3kx + x + 7 = 0.$
8. $2x^2 + k^2x - 4x + 3 = 0.$

Determine k so that the sum of the roots may be 1.

9. $kx^2 - 2kx + x - 2 = 0.$
10. $9x^2 - 4kx + 3x - 4 = 0.$

Determine k so that the product of the roots may be 1.

11. $3kx^2 - 25k + k + 8 = 0.$
12. $2x^5 - 5x + k^2 - 2k - 1 = 0.$

49. Nature of the roots. In Art. 43, we found the two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

to be $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In case a, b, c are real numbers, the numerical character of these roots depends upon the number $b^2 - 4ac$ under the radical sign. An examination of x_1 and x_2 leads at once to the following conclusions:

(1) If $b^2 - 4ac > 0$, the roots are real and unequal.

(2) If $b^2 - 4ac < 0$, the roots are imaginary and unequal.

(3) If $b^2 - 4ac = 0$, the roots are real and equal.

It should be observed that if the coefficients are real and one root is imaginary, then both roots are imaginary.

The quantity $b^2 - 4ac$ is called the **discriminant** of the equation

$$ax^2 + bx + c = 0.$$

50. Sum and product of the roots. If we add together the two roots of $ax^2 + bx + c = 0$, we have

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

If we multiply the two roots together, we have

$$x_1 x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}.$$

Hence:

I. *The sum of the roots of a quadratic equation in x is equal to the coefficient of x with its sign changed, divided by the coefficient of x^2.*

II. *The product of the roots of a quadratic equation in x is equal to the known term divided by the coefficient of x^2.*

EXERCISES

Determine the nature of the roots of the following equations.

1. $x^2 + 11x + 30 = 0$.

2. $x^2 - 8x + 25 = 0$.

3. $x^2 - 16x + 64 = 0$.

4. $2x^2 - 3x - 2 = 0$.

5. $4x^2 - 28x + 49 = 0$.

Determine the real values of k so that the roots of the following equations may be equal.

6. $x^2 + 3kx + k + 7 = 0$.

SOLUTION: In order that the roots of this equation may be equal, we must have $b^2 - 4ac = 9k^2 - 4(k + 7) = 0$. Hence, k must be a solution of the equation $9k^2 - 4k - 28 = 0$, or $k = 2$, or $\frac{-14}{9}$. Substituting these values in the above equation, we get

$$x^2 + 6x + 9 = (x + 3)^2 = 0.$$

$$x^2 - \tfrac{4}{9}x - \tfrac{14}{9} + 7 = \tfrac{1}{9}(9x^2 - 42x + 49) = \tfrac{1}{9}(3x - 7)^2 = 0.$$

7. $x^2 + kx + 4 = 0$.

8. $kx^2 + 6x + 1 = 0$.

9. $4x^2 + 12x + k = 0$.

10. $k^2x^2 + 10x + 1 = 0$.

11. $(k + 1)x^2 + kx + k + 1 = 0$.

12. $x^2 + 12x + 8k = 0$.

13. $(4x + k)^2 = 16x$.

14. $x^2(1 + m^2) + 2kmx + k^2 - r^2 = 0$.

15. $a^2(mx + k)^2 + b^2x^2 = a^2b^2$.

Determine by inspection the sum and product of the roots of the following equations.

16. $7x^2 + 4x - 3 = 0$.

17. $11 - 27x - 18x^2 = 0$.

18. $mx^2 + 2x - 6 = 0$.

19. $8x^2 + 7x - 62 = 0$.

20. $(1 - e^2)x^2 - 2mx + m^2 = 0$.

Determine the value of k in the following equations.

21. $x^2 + kx - 5 = 0$, where one root is -5.

SOLUTION: Let x_1 be the second root. The product of the roots of this equation (II, Art. 50) is -5.

Hence, $\quad -5x_1 = -5$, or $x_1 = 1$,

and, $\quad -5 + x_1 = -k$, or $k = 4$.

22. $x^2 + x - 4k = 0$, where one root is -4.

23. $x^2 - x - k = 0$, where the difference between the roots is 9.

24. $15x^2 + kx - 4 = 0$, where one root is $\frac{2}{3}$.

25. $kx^2 + 6x + 5 = 0$, where one root is 5 times the other.

26. $3x^2 - kx + 14 = 0$, where the quotient of the two roots is $\frac{7}{3}$.

51. Graph of the quadratic function. In Chapter III we have plotted certain quadratic functions. It can be shown, if a is positive and different from zero, that the graph of the function $ax^2 + bx + c$ has the same general characteristics as the curve in Fig. 10. This curve is called a parabola. The real roots of the equation $ax^2 + bx + c = 0$ are given by the abscissas of the points where the curve crosses the X-axis. If the curve has no point in common with the axis, then the roots of the equation are imaginary.

FIG. 10.

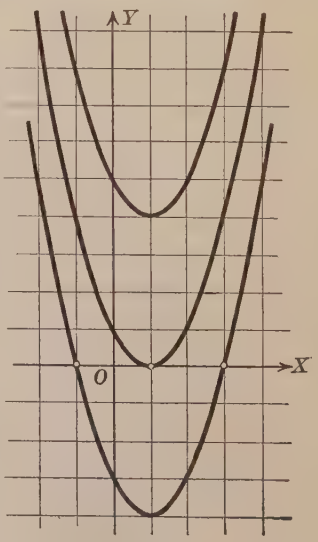

FIG. 11.

For we have shown that every quadratic equation has two roots, real or imaginary. If the curve has no point in common with the X-axis, there is no real root of the equation. Hence both roots are imaginary. If the curve touches the X-axis, both roots of the equation are real and equal. These three cases are shown in Fig. 11, where the graphs of $x^2 - 2x - 3$, $x^2 - 2x + 1$, and $x^2 - 2x + 5$ are given.

EXERCISES

Construct the graphs of the functions in the following equations, and, by measurement, determine the roots if they are real. Calculate the value of the function for at least ten values of x between the limits given. Choose the vertical unit so that the graph will be of convenient proportions for the coordinate paper.

1. $x^2 - 5x + 4 = 0$, from $x = 0$ to $x = 5$.
2. $x^2 + x - 6 = 0$, from $x = -4$ to $x = 3$.
3. $4x^2 + 12x + 5 = 0$, from $x = -4$ to $x = 1$.
4. $x^2 - 3x = 0$, from $x = -1$ to $x = 4$.
5. $x^2 + 2x + 2 = 0$, from $x = -3$ to $x = 2$.
6. $x^2 - 6x + 11 = 0$, from $x = 0$ to $x = 5$.
7. $6 - 3x - x^2 = 0$, from $x = -4$ to $x = 2$.
8. $-3x^2 + 2x - 4 = 0$, from $x = -2$ to $x = +2$.
9. $4 - 5x - x^2 = 0$, from $x = -6$ to $x = 1$.
10. $x^2 - 1 = 0$, from $x = -3$ to $x = +3$.
11. What are the general characteristics of the graph of the function $ax^2 + bx + c$ if a is negative?

PROBLEMS

1. In the course of Steinmetz's solution of the problem of finding the current strength in a divided electric circuit, it is necessary to solve the equation

$$a^2x^2 - as^2 + r^2 = 0$$

for a. His solution is

$$\frac{s^2 \pm q^2}{2x^2},$$

where $q^2 = \sqrt{s^4 - 4r^2x^2}$. Verify the result.

2. If \$10,000 amounts to \$11,130.25 when placed at compound interest for two years, interest being compounded annually, what is the rate of interest?

3. A rectangular court is 25 feet longer than it is wide and it contains 3750 square feet. What are its dimensions?

4. The altitude of a triangle exceeds its base by 48 feet. The area of the triangle is 1387.5 square feet. Find the base and altitude.

5. If a ball is thrown upwards with a velocity v_0, the distance d from the earth to the ball after a given time t is given by the formula

$$d = v_0 t - \tfrac{1}{2} g t^2, \tag{1}$$

where $g = 32.2$. The speed at the time t is given by

$$v_t = v_0 - gt. \tag{2}$$

If the ball is thrown *downwards* with a speed v_0, the above formulas become

$$d = v_0 t + \tfrac{1}{2} g t^2, \tag{3}$$

$$v_t = v_0 + gt. \tag{4}$$

If a ball is thrown upwards with a velocity of 50 feet per second, in what time will it be just 30 feet from the ground? Explain the two answers.

6. How long will it take the ball described in Problem 5 to reach the ground?

[HINT: Put $d = 0$ in formula (1).]

7. At what time is the velocity of the ball zero?

8. How high will the ball rise?

9. How does the time taken in rising to the highest point compare with the whole time that the ball is in the air?

10. How far does the ball rise in the second second?

11. How long will it take a body to fall 400 feet, if it is thrown downward with an initial speed of 50 feet per second?

12. What is the velocity of a falling body at the end of $2\frac{1}{2}$ seconds if the initial velocity be 0?

13. If a body falls from rest, how far will it fall in the fifth second?

14. The edges of a cube are each increased in length one inch. It is found that the volume of the cube is thereby increased 19 cubic inches. What is the length of the edges of the cube?

15. What is the area of a square whose diagonal is one foot longer than a side?

16. What is the area of an equilateral triangle whose side is one foot longer than the altitude?

17. Show that the equation

$$x^2 + bx + c = 0$$

has one positive and one negative solution if c is negative.

18. In joining together two steel boiler plates with a single row of rivets, the distance p between the centers of the rivets is given by the formula

$$p = 0.56 \frac{d^2}{t} + d,$$

where t is the thickness of the plate and d the diameter of the rivet holes. In a boiler the rivets are to be placed $1\frac{1}{2}$ inches apart. If the thickness of the plate is $\frac{1}{4}$ inch, what is the diameter of the rivet holes?

19. Graph on the same sheet the function $2x^2 - x + c$, where c takes the values 1, -3, 0, 10. What effect does changing the constant term in a quadratic function have on the graph?

20. Graph on the same sheet the function $ax^2 - x + 3$, where a takes the values 5, 1, $\frac{1}{2}$, $\frac{1}{10}$. Decreasing the coefficient of x^2 has what effect on the graph? What is the effect on the roots of the quadratic equation $ax^2 - x + 3 = 0$, if a is made to approach 0?

21. If s is the area in square inches of the flat end of a boiler, and t the thickness of the boiler plate in sixteenths of an inch, then the pressure p per square inch which the flat end plate can safely sustain is given by the formula

$$p = \frac{200(t+1)^2}{s-6}.$$

What should be the thickness to the nearest sixteenth of an inch of the boiler plate for the end of a boiler 20 inches in diameter to sustain a pressure of 100 pounds per square inch?

22. In a group of points every point is connected with every other point by a straight line. There are 300 straight lines. How many points are there?

23. A rectangular piece of tin is twice as long as it is wide. From each corner a 2-inch square is cut out, and the ends are turned up so as to make a box whose contents are 60 cubic inches. What are the dimensions of the piece of tin?

24. Let h be the height and t the thickness (in feet) of a rectangular masonry retaining wall. For very sandy soil with a grade angle of 20°, h and t are connected by the equation

$$t^2 + 0.19\,t \cdot h - 0.18\,h^2 = 0.$$

What should be the thickness (to the nearest inch) of a retaining wall four feet high?

25. For loam, the equation in Problem 24 would be

$$t^2 + 0.14\,t \cdot h - 0.13\,h^2 = 0.$$

What should be the thickness of a retaining wall four feet high?

26. A long horizontal pipe is connected with the bottom of a reservoir. If H be the depth of the water in the reservoir in feet, d the diameter of the pipe in inches, L the length of the pipe in feet, and v the velocity of the water in the pipe in feet per second, then according to Cox's formula

$$\frac{Hd}{L} = \frac{4v^2 + 5v - 2}{1200}.$$

Find the velocity of water in a 5-inch pipe, 1000 feet long, connected with a reservoir containing 49 feet of water.

27. The so-called effective area of a chimney is given by

$$E = A - 0.6\sqrt{A},$$

where A is the measured area. Find A when E is 30 square feet.

28. A stone is dropped into a well, and 4 seconds afterward the report of its striking the water is heard. If the velocity of sound is taken at 1190 feet per second, what is the depth of the well? (Use $g = 32.2$. See Problem 5.)

29. The electrical resistance of a wire depends upon the temperature of the wire according to the formula

$$R_t = R_0(1 + at + bt^2),$$

where a and b are constants depending on the material, R_0 is the resistance at 0°, and R_t the resistance at $t°$. For copper wire $a = 0.00387$, $b = 0.00000597$, and $R_0 = 0.02057$. At what temperature is the resistance double that at 0°?

30. The radius of a cylinder is 10 and its height 4. What value can be added to either the radius or to the height, and yet give the same increase in volume?

The following equations occur in some electrical problems.

31. $g = \dfrac{R}{R^2 + X^2}$. Solve for R.

32. $T = \dfrac{a(n - n')}{1 + b(n - n')^2}$. Solve for $(n - n')$.

33. $P = \dfrac{AW(r^2 + x^2)}{r(Rr - Xx)}$. Solve for x.

34. In making war bread a mixture of rye and corn meal was used. From a hundred pounds of rye flour a certain amount was taken and replaced by corn meal. Later, from the mixture the same amount was removed and again replaced by corn meal. The resulting mixture was 16 parts rye to 9 parts corn. What were the proportions in the first mixture?

Locus: All points whose coordinates satisfy the equation and of such points only.

CHAPTER VII

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

52. Type form. The type form of a quadratic equation in two unknowns is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where at least one of the coefficients A, B, or C is not zero. Such an equation is satisfied by an indefinite number of pairs of values of x and y. If the pairs of real values of x and y which satisfy the equation be considered as the coordinates of points, and these points plotted on coordinate paper, they will lie on a curve called the **locus** of the equation.

EXERCISES

Arrange the following equations in the typical form, find values of A, B, C, D, E, and F. Find at least 4 pairs of values of x and y which satisfy each equation.

1. $(x + y)^2 = 3x^2 - y + 2.$

SOLUTION: Written in the typical form this becomes

$$-2x^2 + 2xy + y^2 + y - 2 = 0,$$

where $A = -2,\ B = 2,\ C = 1,\ D = 0,\ E = 1,\ F = -2.$

Substituting $x = 1$ in the equation, there results,

$$y^2 + 3y - 4 = 0,$$

or $$y = 1, \text{ or } -4.$$

Hence, $x = 1,\ y = 1$; $x = 1,\ y = -4$ are two pairs of values satisfying the equation.

Putting $x = 2$ in the equation, we find in the same way two other pairs of values, $x = 2,\ y = \dfrac{-5 + \sqrt{65}}{2}$; $x = 2,\ y = \dfrac{-5 - \sqrt{65}}{2}$, etc.

2. $x^2 - xy + y^2 = 37.$

3. $3x^2 - 2y^2 = 4.$

4. $x^2 + y^2 = 25.$

5. $x^2 + y^2 = (x + y + 1)^2.$

6. $xy = 2x - y + 9.$

7. $(x + y)(3 - x) = (x - y)(2 + y).$

53. Solution of systems of equations involving quadratics. Although there are an indefinite number of pairs of values of x and y which satisfy one quadratic equation, there are not more than four pairs which satisfy two non-equivalent quadratic equations. If these values are real numbers, they are the coordinates of points lying on the locus of each of the given equations. Hence, the real solutions of a pair of simultaneous quadratic equations can be represented graphically by the points of intersection of the loci of the two equations. The general problem of simultaneous quadratics is that of finding pairs of values of x and y which satisfy two equations of the form,

$$\left.\begin{aligned} A_1x^2 + B_1xy + C_1y^2 + D_1x + E_1y + F_1 = 0 \\ A_2x^2 + B_2xy + C_2y^2 + D_2x + E_2y + F_2 = 0 \end{aligned}\right\},$$

where the coefficients may have any values. As illustrated in the following example, this general problem involves the solution of an equation of the fourth degree.

Solve $\left.\begin{aligned} x^2 + y^2 + x - 9 = 0, \\ x^2 + 2y^2 - 3y - 8 = 0. \end{aligned}\right\}$

Subtracting the second from the first, we have

$$-y^2 + 3y + x - 1 = 0,$$

or $$x = 1 - 3y + y^2.$$

Substituting in the second equation, we have

$$(1 - 3y + y^2)^2 + 2y^2 - 3y - 8 = 0,$$

or $$y^4 - 6y^3 + 13y^2 - 9y - 7 = 0.$$

At this stage of our progress in algebra, we cannot solve a general equation of the fourth degree, hence we cannot proceed with the solution of this problem. Although we cannot solve the general problem of simultaneous quadratics, yet there are some types of these equations which can easily be solved. We take up a few of the most important.

Case I. *When each equation is of the form*

$$Ax^2 + Cy^2 + F = 0.$$

If, instead of x and y, we consider x^2 and y^2 as the unknowns, the method of solution is that for linear equations.

EXERCISES

Solve

1. $16x^2 + 27y^2 = 576,$
$x^2 + y^2 = 25.$

SOLUTION: Solving for x^2 and y^2 we have

$$x^2 = \frac{\begin{vmatrix} 576 & 27 \\ 25 & 1 \end{vmatrix}}{\begin{vmatrix} 16 & 27 \\ 1 & 1 \end{vmatrix}} = \frac{-99}{-11} = 9, \qquad y^2 = \frac{\begin{vmatrix} 16 & 576 \\ 1 & 25 \end{vmatrix}}{\begin{vmatrix} 16 & 27 \\ 1 & 1 \end{vmatrix}} = 16.$$

$x = \pm 3.$ $y = \pm 4.$

Hence, we find the following four solutions,

$$(3, 4),\ (-3, 4),\ (3, -4),\ (-3, -4).$$

To show these solutions graphically, we plot the loci of the two equations.

Solving each for y, we have $y = \pm\sqrt{\dfrac{576 - 16x^2}{27}},$

$$y = \pm\sqrt{25 - x^2}.$$

The first equation has for its locus the oval-shaped figure A, B, C, D, called an ellipse, the second, the circle (Fig. 12). The points of intersection represent graphically the four pairs of solutions. If the loci of two equations do not intersect, the solutions of the equations will be found to be imaginary.

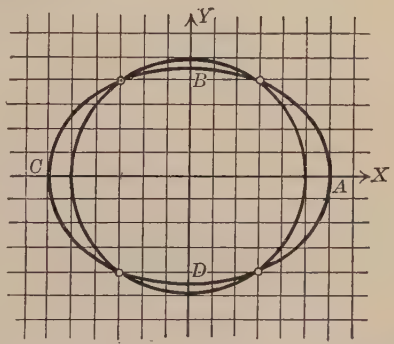

FIG. 12.

Plot and solve:

2. $9x^2 + 25y^2 = 225,$*
$x^2 + y^2 = 16.$

3. $9x^2 + 25y^2 = 225,$
$x^2 + y^2 = 9.$

4. $9x^2 + 25y^2 = 225,$
$x^2 + y^2 = 4.$

5. $9x^2 + 25y^2 = 225,$
$x^2 + y^2 = 25.$

6. $9x^2 + 25y^2 = 225,$
$25x^2 + 9y^2 = 225.$

7. $4x^2 - 9y^2 = 36,$
$x^2 + y^2 = 16.$

Obtain to two significant figures the solutions of the following:

8. $4.3x^2 + 9.1y^2 = 89,$
$x^2 + y^2 = 9.8.$

9. $1.9x^2 + 0.21y^2 = 3.6,$
$x^2 - y^2 = -4.1.$

---

* The graph of $9x^2 + 25y^2 = 225$ is an ellipse of breadth 10 and height 6. Its position with respect to the axes is similar to that of the ellipse in Fig. 12. The graph of $x^2 + y^2 = 16$ is a circle of radius 4, and center at the intersection of the axes.

Case II. *When one of the equations is linear.*

EXERCISES

Plot and solve:

1. $x^2 + y^2 - 8x - 4y - 5 = 0,$
$3x + 4y = 5.$

Solution: From the linear equation,

$$x = \frac{5 - 4y}{3}.$$

Substituting in the quadratic equation and reducing, we have

$$5y^2 + 4y - 28 = 0.$$

Solving, we obtain $y = 2$, or $-\frac{14}{5}$.

Substituting these values in the linear equation, we find

$$x = -1, \text{ or } \frac{27}{5}.$$

The solutions are then $(-1, 2)$, $(\frac{27}{5}, -\frac{14}{5})$.

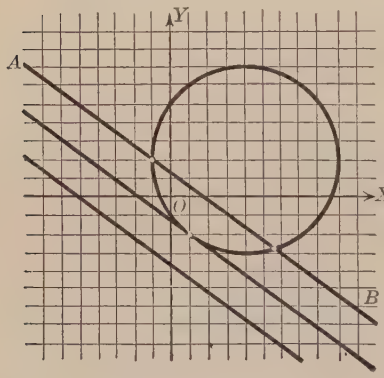

Fig. 13.

The loci of these two equations are shown in Fig. 13. The circle with its center at the point (4, 2) and radius 5 is the locus of the first equation, while the straight line AB is the locus of the second equation.*

2. $x^2 + y^2 - 8x - 4y - 5 = 0,$
$3x + 4y + 5 = 0.$

3. $x^2 + y^2 - 8x - 4y - 5 = 0,$
$3x + 4y + 15 = 0.$

4. $x^2 + y^2 = 58,$
$x + y = 10.$

5. $xy = 36,$
$x + y = 13.$

6. $xy = 40,$
$x - y = 3.$

7. $xy - 5x = 1,$
$7x - y = 1.$

8. $\frac{x}{4} + \frac{y}{5} = 3,$
$(x + y)^2 = 200 - x.$

9. $y^2 - 4x = 0,$
$x - y + 1 = 0.$

10. $(3x + y)(y + 2) = (x + 5)(3x - y),$
$2x - y = 1.$

11. $x + \frac{1}{3} = \frac{2x + y}{3},$
$2(x + y) = x(4x - y).$

12. $0.1y + 0.125x = y - x,$
$y - 0.5x = 0.75xy - 3x.$

13. $0.3x + 0.125y = 3x - y,$
$3x - 0.5y = 2.25xy + 3y.$

* The remaining two lines of the figure belong to the next two exercises. In Ex. 2, where the line touches (is tangent to) the circle, only one pair of numbers satisfies the system, and the solutions are said to be equal.

Obtain to two significant figures the solutions of the following:

14. $xy = -0.24$,
$3.1x - 0.63y = 4.3$.

15. $y^2 - 4.1x = 0.38$,
$x - 0.37y = 0.29$.

Find the values of a, b, c, or r in the following exercises so that the straight line which is the locus of the first degree equation

(1) cuts the other locus in two distinct points,

(2) is tangent to the curve,

(3) fails to meet the curve.

16.
$$x^2 + y^2 = r^2,$$
$$3x + 4y = 5.$$

The locus of the first equation is a circle with center at the origin and a radius equal to r.

SOLUTION: From the second equation, we have

$$x = \frac{5 - 4y}{3}.$$

Substituting in the first, we find

$$25y^2 - 40y + 25 - 9r^2 = 0.$$

Solving for y, we obtain, $$y = \frac{40 \pm 30\sqrt{r^2 - 1}}{50}.$$

If y is real, $r^2 - 1$ must be equal to or greater than zero.

Furthermore, if r is any number greater than 1, the two loci intersect in real and distinct points.

If $r = 1$, there is only one value for y, and the line is tangent to the circle. If $r < 1$, the line does not intersect the circle.

17. $x^2 + y^2 = r^2$,
$x + y = 10$.

18. $x^2 + y^2 = 25$,
$3x + 4y = c$.

19. $x^2 + y^2 = 25$,
$ax + 6y = 1$.

20. $x^2 + y^2 = 25$,
$7x - by = 3$.

21. For what values of b in terms of r and m does the system of equations

$$y = mx + b,$$
$$x^2 + y^2 = r^2$$

have equal solutions?

22. Determine the relation between a, b, and k such that the system

$$y = mx + k,$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has equal solutions.

Case III. *When all the terms which contain the unknowns are of the second degree.*

EXERCISES

Solve the following pairs of equations.

1. $x^2 + 3xy = 28,$
$x^2 + y^2 = 20.$

Solution: Let $y = mx$ and substitute in both equations.
From the first equation, we have

$$x^2 + 3mx^2 = 28,$$

whence $$x^2 = \frac{28}{1 + 3m}.$$

From the second equation, we have

$$x^2 + m^2x^2 = 20,$$

whence $$x^2 = \frac{20}{1 + m^2}.$$

Equating these values of x^2, we obtain

$$\frac{28}{1 + 3m} = \frac{20}{1 + m^2}.$$

Clearing of fractions and reducing, we obtain

$$7m^2 - 15m + 2 = 0,$$

or $$m = 2, \text{ or } \tfrac{1}{7}.$$

Substituting these values of m in $x^2 = \frac{28}{1 + 3m}$,

we find, for $m = 2$, $$x^2 = 4, x = \pm 2,$$
$$y = mx = \pm 4.$$

For $m = \frac{1}{7}$, we find $$x^2 = \tfrac{98}{5}, x = \pm \tfrac{7}{5}\sqrt{10} = \pm 4.43^+,$$
$$y = mx = \pm \tfrac{1}{5}\sqrt{10} = \pm 0.63^+.$$

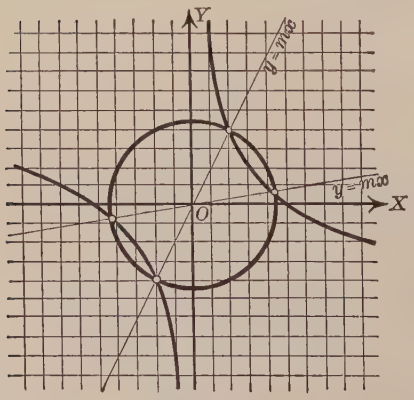

Fig. 14.

The solutions are therefore

$(2, 4), (-2, -4), (\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}), (-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10})$.

The loci of the two equations of this exercise are shown in Fig. 14. The geometrical interpretation of the substitution $y = mx$ is also shown in the figure.

2. $x^2 + 3xy = 28,$
$4y^2 + xy = 8.$

3. $x^2 + xy + y^2 = 6,$
$x^2 + y^2 = 12.$

4. $2y^2 - 4xy + 3x^2 = 17,$
$y^2 - x^2 = 16.$

5. $x^2 + y^2 = 65,$
$xy = 28.$

6. $x^2 + xy + 2y^2 = 74,$
$2x^2 + 2xy + y^2 = 73.$

7. $x^2 - 4y^2 = 9,$
$xy + 2y^2 = 3.$

8. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3},$
$x^2 + y^2 = 45.$

9. $x^2 - xy + y^2 = 21.$
$y^2 - 2xy + 15 = 0.$

10. $4a^2 - 2ab = b^2 - 16,$
$5a^2 = 7ab - 36.$

11. $x^2 + xy = 4,$
$y^2 + xy = 1.$

Find to two significant figures the solutions of the following:

12. $x^2 + 1.6y^2 = 8.3,$
$xy = 2.3.$

13. $x^2 + 0.12xy = 104,$
$y^2 + 1.4xy = 21.$

CASE IV. *When the equations are symmetrical.**

The typical form of a symmetrical quadratic equation in two unknowns is

$$A(x^2 + y^2) + Bxy + D(x + y) + F = 0.$$

EXERCISES

Solve the following systems of equations:

1.
$$x^2 + y^2 + x + y = 8, \quad (1)$$
$$xy + x + y = 5. \quad (2)$$

SOLUTION: Let $x = u + v$, $y = u - v$. Substituting in the two equations we obtain after reductions

$$u^2 + v^2 + u = 4, \quad (3)$$
$$u^2 - v^2 + 2u = 5. \quad (4)$$

Eliminating v^2 by adding we obtain an equation in u,

$$2u^2 + 3u = 9,$$

from which $u = \frac{3}{2}$ or -3.

The four solutions of (3) and (4) are then

$$(\tfrac{3}{2}, \tfrac{1}{2}),\ (\tfrac{3}{2}, -\tfrac{1}{2}),\ (-3, i\sqrt{2}),\ (-3, -i\sqrt{2}).$$

From the first pair

$$x = u + v = \tfrac{3}{2} + \tfrac{1}{2} = 2,$$
$$y = u - v = \tfrac{3}{2} - \tfrac{1}{2} = 1.$$

In the same way we find from the other pairs,

$$\left.\begin{matrix} x = 1 \\ y = 2 \end{matrix}\right\},\quad \left.\begin{matrix} x = -3 + i\sqrt{2} \\ y = -3 - i\sqrt{2} \end{matrix}\right\},\quad \left.\begin{matrix} x = -3 - i\sqrt{2} \\ y = -3 + i\sqrt{2} \end{matrix}\right\}.$$

---

* An equation is said to be symmetrical with respect to x and y whenever interchanging x and y leaves the equation unchanged.

2. $x + y - 2xy = 5,$
$x^2 + 6xy + y^2 - 4x - 4y = 5.$

3. $3x^2 + 3y^2 = 8(x + y) - 1,$
$xy = x + y + 1.$

4. $x^2 + y^2 + xy + x + y = 17,$
$x^2 + y^2 - 3xy + 2x + 2y = 9.$

5. $3x(1 + x) + 3y(1 + y) = 54,$
$2x + 2y + xy = 16.$

6. $x^2 + y^2 = 13,$
$xy = 6.$

Many systems of equations of degree higher than two, and systems containing three or more unknowns may be solved by combinations and variations of the above methods, but these four cases do not by any means include all the simultaneous equations whose solution can be reduced to the solution of the quadratic. Usually the solution of such a system is in the nature of a puzzle for which no special rules can be given. Whatever method be used it must be kept in mind that the ultimate test of a solution is substitution in the given equations. Many equations coming under the cases mentioned here may be solved more easily by other special methods. For example, Exercise 6 under Case IV may be done as follows:

Given $x^2 + y^2 = 13,$
$xy = 6.$

Multiply the second equation by 2, add this result to and subtract it from the first equation, and thus obtain the system

$$x^2 + 2xy + y^2 = 25,$$
$$x^2 - 2xy + y^2 = 1.$$

Extracting the square root of each, we have

$$x + y = \pm 5,$$
$$x - y = \pm 1.$$

From these equations we find the same four results which were obtained by the general method for solving symmetrical equations.

EXERCISES AND PROBLEMS

1. $\frac{x}{y} + \frac{y}{x} = \frac{10}{3},$
$4x - 7y = 5.$

2. $3x - \frac{1}{y} = 4,$
$9x^2 + \frac{1}{y^2} = 40.$

3. $\frac{1}{x} - \frac{1}{y} = 5,$
$\frac{1}{x^2} + \frac{1}{y^2} = 233.$

4. $x + y + 2y^2 = 11,$
$3x - 2y - 2y^2 + 9 = 0.$

5. $xy - x = 12,$
$xy + 3y = 35.$

6. $2y^2 - 4xy + 3x^2 = 17,$
$y^2 - x^2 = 16.$

7. $(x+1)(y+2) = 28,$
$(x+3)(y+4) = 54.$

8. $x^2 + y^2 = 8y + 1 - 2xy,$
$2(x+y)^2 = 10x + 9y + 3.$

9. $x(y-4) = 14,$
$y(x+1) = 33.$

10. $3(x^2 - y^2) = 2x + 17,$
$x^2 - y^2 + x + y = 18.$

11. $(x-1)(y+5) = 80,$
$(x+4)(y-2) = 39.$

12. $(x+y)^2 + (x+y) = 12,$
$3x^2 + y^2 = x + y + 4.$

13. $x^2 - 4xy + 3y^2 = 0,$
$x^2 + y^2 = 5(y+6).$

14. $2(x+1)(y-4) = y + 1,$
$(3x-1)(2y-9) = 5.$

15. $(x - 4y - 3)(x + y) = 0,$
$x^2 + y^2 = 22 + 4xy.$

16. $x^2 - xy - 2y^2 = 0,$
$x^2 + 2y^2 - 3x = 12.$

17. $(x-y)^2 + (x-y) = 6,$
$x^2 = y(2x+1) + 2x - 18.$

HINT: Solve the first equation for $(x-y)$.

18. $x^3 + y^3 = 28,$
$x + y = 4.$

HINT: Divide the first equation by the second.

19. $x^3 - y^3 = 19,$
$x - y = 1.$

20. $x^3 - y^3 = 63,$
$x^2 + xy + y^2 = 21.$

21. $x^2 + 2xy + y^2 = (x+y)(3x+1),$
$x^2 - y^2 + 2xy + 1 = 0.$

22. $\sqrt{x} - \sqrt{y} = \frac{1}{3}(x-y),$
$xy = 36.$

23. $x^2 - y\sqrt{xy} = 14,$
$y^2 - x\sqrt{xy} = -7.$

24. $x^4 - y + \sqrt{x^2 - y} = 20,$
$x^2 - y^2 = 544.$

25. $x^2 - xy + y^2 = 3a^2,$
$x + y = 3(x - y).$

26. $x^2 + y^2 = a^2,$
$\sqrt{x} + \sqrt{y} = \sqrt{a}.$

27. $x^2 + y^2 = 13,$
$y^2 + z^2 = 25,$
$z^2 + x^2 = 20.$

28. $x^2 + 2y^2 - z^2 = 5,$
$2x + y + z = 6,$
$x + 4y - z = 5.$

29. $x(y+z) = 8,$
$y(z+x) = 18,$
$z(x+y) = 20.$

30. $x + y = 11,$
$z + u = 10,$
$xy = zu,$
$x^2 + y^2 + z^2 + u^2 = 125.$

31. $xy + zu = 17,$
$xz + yu = 13,$
$xu + yz = 11,$
$x + y + z + u = 11.$

32. $x^2 + y^2 + z^2 + u^2 + v^2 = 19,$
$x + y + z = u + v - 1,$
$x + v = z + u,$
$x + y + v = 5,$
$y + z = v.$

Eliminate x and y from the following equations.

33. $x^2 + y^2 = a^2$,
$x + y = b$,
$x - y = c$.

34. $x^2 + y^2 = a^2$,
$x^2 - y^2 = b^2$,
$xy = c^2$.

35. $x^2 + xy = a^2$,
$y^2 + xy = b^2$,
$x^2 + y^2 = c^2$.

36. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$,
$xy = ab$,
$a^2x^2 + b^2y^2 = c^4$.

37. A circular track is built so that the width of the track is $\frac{1}{50}$ of the inside diameter. After construction it was found that the area of the track was 2564 square yards. What are the inside and outside lengths of the track?

38. The fence around a rectangular field is 1400 feet long. The diagonal of the field is 500 feet long. What are the dimensions of the field?

39. The diagonal of a rectangular parallelopiped is 14 inches long. The sum of the three dimensions is 22. The reciprocal of one dimension is one half the sum of the reciprocals of the other two. What are the dimensions of the solid?

40. A father divided $1000 between his two sons and kept it for them at simple interest until called for. At the end of 3 years, one son called for all the money due him and received $665.50. At the end of 4 years the other son received $576 as his share. How was the money originally divided and what rate of interest did the father pay?

41. A few days after the outbreak of the war a 25-pound bag of sugar cost the retailer $77\frac{1}{2}$ cents more than it did just before the outbreak. For $165 a grocer received 1550 pounds less sugar after the outbreak than he would have received before for the same amount. What was the price before and after the outbreak of the war?

42. A silver wire 1000 millimeters long is to be covered with a layer of gold until the diameter of the wire is increased one tenth. It is found that 4 cubic millimeters of gold is needed. What is the diameter of the wire?

43. A road between two towns is 33 miles long. At eight o'clock, from each of the two towns, a traveler starts toward the other and walks at a uniform speed. At eleven o'clock they meet, but one traveler arrives at his destination 1 hour and 6 minutes earlier than the other. How many minutes are needed for each to walk 1 mile?

44. A man divides a tract of land into city lots. He sells the lots all at the same price and realizes $4800. If the number of lots had been one greater, and the price $8 per lot cheaper, he would have received the same amount of money. How many lots were there and what was the price per lot?

45. A circular cylinder is inscribed in a sphere of radius 10. The total surface of the cylinder is half the surface of the sphere. What is the radius and what is the altitude of the cylinder?

46. The diagonals of the three faces of a rectangular parallelopiped which meet in a vertex of the solid are 10, 12, 14, respectively. What is the volume of the solid?

47. The floor of a rectangular room contains 273 square feet, one wall 189 square feet, and the adjacent wall 117 square feet. What are the dimensions of the room?

48. Psychologists assert that the rectangle most pleasing to the human eye is that in which the sum of the two dimensions is to the longer as the longer is to the shorter. If the area of a page of this algebra remains unchanged, what should its dimensions be?

49. An aëroplane, flying 75 miles per hour and following a long straight road, passed an automobile going in the opposite direction. One hour later it overtook a second automobile. The automobiles passed each other when the aëroplane was 100 miles away. If both automobiles travel with the same speed, how far apart were they when the aëroplane passed the second one and what was their speed?

50. Two students attempt to solve a problem that reduces to a quadratic equation. One in reducing has made a mistake only in the constant term of the equation, and finds 8 and 2 for the roots. The other makes a mistake only in the coefficient of the first degree term, and finds -9 and -1 for roots. What was the quadratic equation?

51. The radius of the front wheel of a carriage is 6 inches less than that of the rear wheel. If the front wheel makes 80 more revolutions than the rear wheel in going a mile, what is the circumference of each wheel to the nearest inch?

52. Two polygons have together 12 sides and 19 diagonals. How many sides has each?

53. A military ambulance traveling 12 miles per hour sent on ahead a motor cyclist messenger who could travel with twice the speed of the ambulance. A half-hour later it was found necessary to revise the message, and a second motorcyclist was sent to overtake the first and to give him the new message. The second messenger returned to the ambulance in forty-five minutes. What was his average speed during that time?

CHAPTER VIII

INEQUALITIES

54. Definition. The expressions "a is greater than b" ($a > b$) and "c is less than d" ($c < d$), when a, b, c, d, are real numbers, mean that $a - b$ is a positive number and $c - d$ is a negative number. Such expressions are called **inequalities.** Two inequalities $a > b$, $c > d$, which have the signs pointing in the same direction, are said to be *alike in sense.* If the signs point in opposite directions, as $a > b$, $c < d$, they are said to be *different in sense.* The expression $a \leqq b$ is read "a is less than or equal to b," and $a \geqq b$ is read "a is greater than or equal to b."

55. Absolute and conditional inequalities. We have seen that there are two kinds of equalities, identical and conditional equalities. Corresponding to these there are two kinds of inequalities. An inequality such as $a^2 + b^2 > -1$, which is valid for all real values of a and b, is called an **absolute** inequality; while an inequality such as $x - 4 > 0$, which holds only when x is greater than 4, is called a **conditional** inequality. In a conditional inequality the letters cannot take all real values.

56. Elementary principles. The following elementary principles, which follow at once from the definition of an inequality, must be observed in dealing with inequalities.

I. *The sense of an inequality is not changed if both sides are increased or decreased by the same number. In particular, the sense is not changed if we transpose a term, changing its sign.*

Let $a > b.$

Then $a - b = n,$ where n is positive,

and $a + k - b - k = n,$

or $(a + k) - (b + k) = n.$

Hence, $a + k > b + k.$

II. *The sense of an inequality is not changed if both sides are multiplied or divided by the same positive number.*

III. *The sense of an inequality is reversed if both sides are multiplied or divided by the same negative number.*

The proofs of II and III are very similar to the proof of I.

EXERCISES

1. If a and b are not equal, show that $a^2 + b^2 > 2ab$.

SOLUTION: $(a-b)^2 > 0$, since the square of any real number is positive.
That is, $$a^2 - 2ab + b^2 > 0.$$
By Principle I, $$a^2 - 2ab + b^2 + 2ab > 0 + 2ab,$$
or $$a^2 + b^2 > 2ab.$$

2. Show that $\frac{a+b}{2} > \sqrt{ab}$, if a and b are positive and unequal.

Show that the following inequalities subsist, the letters representing distinct positive numbers.

3. $\frac{a+b}{2} > \frac{2ab}{a+b}$.

4. $a^2 + b^2 + c^2 > ab + ac + bc$.

5. $\frac{a^3+b^3}{a^2b^2} > \frac{a+b}{ab}$.

6. $a + \frac{1}{a} > 2$, if $a \neq 1$.

7. $a^2 + 3b^2 > 2b(a+b)$.

8. $\frac{a}{b} + \frac{b}{a} > 2$.

9. If b, d, and f are positive, and $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, show that $\frac{a+c+e}{b+d+f}$ lies between $\frac{a}{b}$ and $\frac{e}{f}$.

10. Given $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, show that $ab + cd < 1$.

11. Show that if we denote by $|a|$,* $|b|$, etc. the numerical values of a, b, etc., then

$$|a+b| \leqq |a| + |b|,$$
$$|a-b| \leqq |a| + |b|,$$
$$|a-b| \geqq |a| - |b|.$$

12. Given
$$a < a_1 < A, \quad b_1 > 0,$$
$$a < a_2 < A, \quad b_2 > 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a < a_n < A, \quad b_n > 0,$$
show that
$$a(b_1 + b_2 + \cdots + b_n) < a_1b_1 + a_2b_2 + \cdots + a_nb_n < A(b_1 + b_2 + \cdots + b_n).$$

---

* The expression $|a|$ is often read "absolute value of a," meaning the numerical value without regard to sign.

57. Conditional inequalities. By transposing terms every inequality may be reduced to an inequality of the form $P > 0$, or $P < 0$. If one or both sides involves a variable, say x, it can be put in one of the two forms $f(x) > 0$, or $f(x) < 0$. In this connection the most important problem is to find the range of values of the variable for which the inequality holds. In the case of linear inequalities the solution is easy. Thus, to find the values of x for which the inequality

$$3x + 19 > 12 - x$$

holds, all the terms can be transposed to the left-hand side, and there results

$$4x + 7 > 0.$$

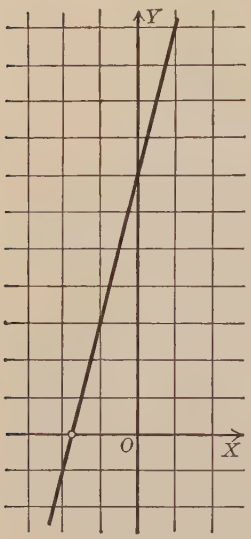

Fig. 15.

Hence the inequality in question holds only for

$$x > -\tfrac{7}{4}.$$

Graphically,

$$3x + 19 > 12 - x$$

for those values of x for which the graph of the function

$$3x + 19 - 12 + x \equiv 4x + 7$$

lies above the X-axis (Fig. 15).

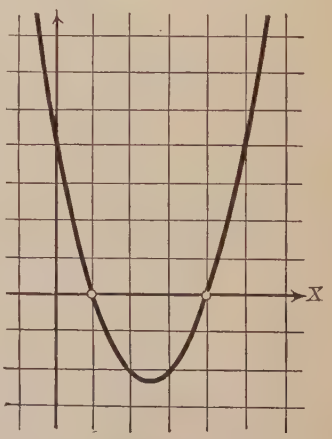

Fig. 16.

The graph is of great service in determining the values of x for which one function of x is greater or less than another function. Thus, to find the range of values of x for which

$$2x^2 - 3x + 8 > x^2 + 2x + 4,$$

we transpose all terms to one side and have

$$x^2 - 5x + 4 > 0.$$

The graph of this function is shown in Fig. 16. It crosses the X-axis at 1 and 4, and for $x > 4$, or $x < 1$, the function $x^2 - 5x + 4$ is positive; while for $4 > x > 1$, it is negative; hence,

$$2x^2 - 3x + 8 > x^2 + 2x + 4$$

for $x > 4$, and $x < 1$, while

$$2x^2 - 3x + 8 < x^2 + 2x + 4$$

for $4 > x > 1$.

EXERCISES

For what values of x do the following inequalities hold ?

1. $7x - 3 > 32$.

2. $\begin{vmatrix} x & 3 \\ 1 & 2 \end{vmatrix} > 0$.

3. $\begin{vmatrix} x & 3 \\ 1 & 2 \end{vmatrix} > \begin{vmatrix} 4 & 3 \\ 7 & x \end{vmatrix}$.

4. $x^2 < 4$.

5. $3x^2 - 11x > 4$.

6. $x + \dfrac{1}{x} > 2$.

7. $(x-5)(x-2) > (x-1)(x-3)$.

8. $2x^2 - 3x - 10 > x^2$.

9. $\dfrac{2x-5}{x+8} < 0$.

10. $\dfrac{a-x}{a+x} < \dfrac{a^2-x^2}{a^2+x^2}$.

11. $ax + b > 0$.

12. $x^2 - 4x + 3 \begin{cases} > 0, \\ = 0, \\ < 0. \end{cases}$

13. $ax^2 + bx + c \begin{cases} < 0, \\ = 0, \\ < 0. \end{cases}$

14. $\dfrac{x+5}{x-3} < 0\,;\ > 0$.

15. $\begin{vmatrix} x & 3 & -1 \\ 2 & 1 & -2 \\ 2 & 3 & x \end{vmatrix} > 0$.

16. $\begin{vmatrix} x & 1 & 4 \\ 3 & x & -5 \\ 4 & 2 & 3 \end{vmatrix} > \begin{vmatrix} -1 & x & 1 \\ 1 & 3 & 2 \\ x & -3 & 2 \end{vmatrix}$.

17. Given $a_1 > a_2 > a_3 > \cdots > a_n$, and $a_r > x > a_{r+1}$; show that

$$(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n)$$

represents a positive number when r is even, and a negative number when r is an odd number.

18. Show that $(x - a_1)(x - a_2)^2$ is positive if $x > a_1$, and negative if $x < a_1$.

CHAPTER IX

MATHEMATICAL INDUCTION

58. General statement. Many important theorems in algebra can be proved by a method called **mathematical induction.** The method may be best explained after applying it to a simple example.

Let it be required to show by mathematical induction that the sum of the first n odd numbers is the square of n. It should be noted that the nth odd number is $2n-1$. We have then to prove

$$1+3+5+\cdots+(2n-1)=n^2$$

for all positive integral values of n.

We note that the theorem is true for $n=1$ and $n=2$. Assume it to be true for $n=r$. We have then

$$1+3+5+\cdots+(2r-1)=r^2. \qquad (1)$$

Adding the next odd number, that is, $2r+1$ to both sides of (1), we obtain

$$1+3+5+\cdots+(2r-1)+(2r+1)=r^2+2r+1=(r+1)^2, \qquad (2)$$

which states that the sum of the first $r+1$ odd numbers is $(r+1)^2$. Hence the theorem is true for $n=r+1$, *if it is true for* $n=r$. We know it is true for $n=2$, hence it is true for $n=2+1=3$. Since it is true for $n=3$, it is true for $n=4$ and so on. Therefore, the sum of the first n odd numbers is the square of n.

In general, an argument by mathematical induction consists of two necessary parts.

First. To show by mere verification that the principle in question is true for some particular case, preferably for $n = 1$, or $n = 2$.

Second. To show that if it is true for the case $n = r$, it is true for the case $n = r + 1$.

It is no proof simply to show that a theorem is true in a number of cases. For example the above theorem is not proved by showing that it is true for $n = 1$, $n = 2$, $n = 3$, $n = 4$ and so on for a definite number of cases. The second part of the proof is necessary.

A celebrated example illustrating this point is the expression $n^2 - n + 41$. From the table we see that $n^2 - n + 41$ is a prime

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
$n^2 - n + 41$	41	43	47	53	61	71	83	97	113	131	151	173

number for all integral values of n up to 12. The table could be continued up to $n = 40$ and the lower row would still contain nothing but prime numbers. However we have no proof that $n^2 - n + 41$ is prime for all integral values of n. To prove this it is necessary to take the second step in the proof by mathematical induction, *i.e.* to prove that if $r^2 - r + 41$ is prime, then $(r + 1)^2 - (r + 1) + 41$ is prime. But this is impossible. In fact, when $n = 41$, we have

$$n^2 - n + 41 = 1681 = 41^2,$$

a number which is not prime.

Again, it is no proof simply to show that if a statement is true for $n = r$ it is true for $n = r + 1$. For example, assuming that the sum of the first r even numbers is an odd number it follows that the sum of the first $r + 1$ even numbers is odd. Though the second part of the proof of this statement by mathematical induction can be correctly presented, we know the statement to be false. The first part of the proof is lacking.

In a proof by mathematical induction both parts are necessary and are of equal importance.

EXERCISES

Prove by mathematical induction.

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

SOLUTION: For $n = 1$, we have

$$2 = 1 \cdot 2.$$

Hence, the statement, $2 + 4 + 6 + \cdots + 2n = n(n + 1)$, is true for $n = 1$, and the first part of the proof is satisfied. Suppose it is true for r numbers. That is, assume that

$$2 + 4 + 6 + \cdots + 2r = r(r + 1). \tag{1}$$

Adding the $(r + 1)$th term to both sides,

$$2 + 4 + 6 + \cdots + 2r + (2r + 2) = r(r + 1) + 2r + 2 = (r + 1)(r + 2). \tag{2}$$

By comparing (1) and (2), it is seen that (2) can be obtained from (1) by replacing r by $r + 1$. That is, if the formula holds for r integers, it holds for $r + 1$ integers. But we know the formula is true for $n = 1$, hence, it is true for $n = 2$, and so on. Hence, for any integer n,

$$2 + 4 + 6 + \cdots + 2n = n(n + 1).$$

2. $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$.

3. $3 + 6 + 9 + 12 + \cdots + 3n = \frac{3n(n + 1)}{2}$.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

5. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(n + 1)(2n + 1)}{3}$.

6. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$ to n terms $= \frac{n}{n + 1}$.

7. $x - y$ is a factor of $x^n - y^n$ if n is any positive integer.

HINT: By verification we see that

$$x^{r+1} - y^{r+1} \equiv x(x^r - y^r) + y^r(x - y).$$

If $x - y$ is a factor of $x^r - y^r$, then it is a factor of $x^{r+1} - y^{r+1}$.

8. $x^{2n} - y^{2n}$ is divisible by $x + y$ if n is any positive integer.

9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4} = (1 + 2 + 3 + \cdots + n)^2$.

10. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

11. $\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}$.

59. Meaning of $r!$. The symbol $r!$, read "factorial r,"* is used to indicate the product $1 \cdot 2 \cdot 3 \cdots r$. Thus, $3! = 1 \cdot 2 \cdot 3 = 6$; $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$.

* The symbol $\lfloor r$ is often used to represent factorial r.

EXERCISES

Evaluate the following expressions:

1. $\frac{9!}{6!}$. 2. $\frac{8!}{7!}$. 3. $\frac{3!\,5!}{4!}$. 4. $\frac{3!+5!}{4!}$.

5. $\frac{9!}{3!\,6!}$. 6. Prove $\frac{r!}{(r-1)!} = r$. 7. $\frac{r!}{(r-2)!} = ?$

60. Binomial theorem; positive integral exponents. By multiplication, we find $(a+x)^2 = a^2 + 2ax + x^2$.

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 = a^3 + 3a^2x + \frac{3\cdot 2}{2!}ax^2 + x^3.$$

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$
$$= a^4 + 4a^3x + \frac{4\cdot 3}{2!}a^2x^2 + \frac{4\cdot 3\cdot 2}{3!}ax^3 + x^4.$$

If n represents the exponent of the binomial in any one of the above three cases, we notice:

(1) The first term is a^n.

(2) The second term is $na^{n-1}x$.

(3) The exponents of a decrease by unity from term to term while the exponents of x increase by unity.

(4) If in any term the coefficient be multiplied by the exponent of a and divided by the exponent of x increased by unity, the result is the coefficient of the next term.

For $n < 5$, we may then write

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1} + \cdots + x^n.$$

Here the question naturally occurs: Does the expansion hold for $n \geqq 5$? It can be shown by mathematical induction that it holds for any positive integral value of n.

Assume

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2!}a^{m-2}x^2 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-r+2)}{(r-1)!}a^{m-r+1}x^{r-1} + \cdots + x^m.$$

Multiply both members of this assumed equality by $a + x$, and we obtain

$$(a+x)^{m+1} =$$
$$a^{m+1} + ma^m x + \cdots + \frac{m(m-1)\cdots(m-r+2)}{(r-1)!}a^{m-r+2}x^{r-1} + \cdots + ax^m$$
$$+ a^m x + \cdots + \frac{m(m-1)\cdots(m-r+3)}{(r-2)!}a^{m-r+2}x^{r-1} + \cdots + max^m + x^{m+1}$$
$$= a^{m+1} + (m+1)a^m x + \cdots + \frac{(m+1)m\cdots(m-r+3)}{(r-1)!}a^{m-r+2}x^{r-1} + \cdots$$
$$+ (m+1)ax^m + x^{m+1}$$

This expansion is the same that would be obtained by substituting $m + 1$ for m in the expansion of $(a + x)^m$. Hence, if the expansion is true for $n = m$, it is true for $n = m + 1$. Since we know it is true for $n = 2$, it is true for $n = 3$, and so on. Hence, when n is any positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1} + \cdots + x^n.$$

This expansion of a binomial is called the **binomial theorem.**

In the expansion of $(a + x)^n$, the rth term is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1},$$

which may also be written

$$\frac{n!}{(r-1)!\,(n-r+1)!}a^{n-r+1}x^{r-1}.$$

The term involving x^r is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}x^r = \frac{n!}{r!\,(n-r)!}a^{n-r}x^r.$$

Each of these terms is sometimes called the general term of the binomial expansion.

In this chapter the exponent n in the binomial expansion is limited to positive integral values, but no assumption has been made with regard to a or x, so we are at liberty to use the expansion no matter what sort of numbers a and x may be. Thus, in

$(2b - 4r^2)$, $a = 2b$ and $x = -4r^2$. In a later chapter it will be shown that the expansion may be interpreted so as to hold when n is a negative or fractional number, but in that case the number x must lie between $-a$ and $+a$.

EXERCISES

Expand

1. $(2x - 3y^3)^4$.

SOLUTION:

$$(2x - 3y^3)^4 = (2x)^4 + 4(2x)^3(-3y^3) + 6(2x)^2(-3y^3)^2 + 4(2x)(-3y^3)^3 + (-3y^3)^4 = 16x^4 - 96x^3y^3 + 216x^2y^6 - 216xy^9 + 81y^{12}.$$

2. $(a + x)^6$.

3. $(a - x)^6$.

4. $(a + b)^7$.

5. $(2 - x)^5$.

6. $(\frac{1}{2} + 2a)^4$.

7. $(1 + a)^5$.

8. $(x - 3y^2)^5$.

9. $(a + \sqrt{b})^6$.

10. $\left(x + \frac{1}{x}\right)^4$.

11. $\left(1 + \frac{1}{e}\right)^6 - \left(1 - \frac{1}{e}\right)^6$.

12. $\begin{vmatrix} a & 1 \\ 2 & b \end{vmatrix}^4$.

13. $(\sqrt{x} - \sqrt{y})^8$.

14. $\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)^6$.

15. $(a^{\frac{3}{2}} + a^{\frac{2}{3}})^6$.

16. $(x^{-1} - y^{\frac{1}{3}})^4$.

17. $(3 - 1)^5$.

18. $\left(\frac{a}{\sqrt{b}} - \frac{b}{\sqrt{x}}\right)^6$.

19. $(a + b + c)^3$.

HINT: Consider $(a + b)$ as representing one number.

20. $\left(\sqrt{x} + 2 + \frac{1}{x}\right)^3$.

21. $\left(\frac{x^2}{3} - \frac{2}{x^3} + 1\right)^4$.

22. Find the seventh term in the expansion of $(x^{\frac{1}{2}} - 2y^2)^{11}$.

SOLUTION: The rth term is given by the expression

$$\frac{n(n-1)\cdots(n-r+2)}{(r-1)!} a^{n-r+1} x^{r-1}.$$

Here $n = 11$, $r = 7$, $a = x^{\frac{1}{2}}$, $x = -2y^2$, $n - r + 2 = 6$.

Substituting these in the expression for the rth term, we have

$$\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{6!} (x^{\frac{1}{2}})^5 (-2y^2)^6 = 29568 x^{\frac{5}{2}} y^{12}.$$

23. Find the fourth term of $(a - 4b)^{12}$.

24. Find the eleventh term of $(2x - y)^{17}$.

25. Find the middle term of $(x^2 + 3y^2)^8$.

26. Find the fourteenth term of $(a + b)^{18}$.

27. Find the eighth term of $\begin{vmatrix} x & a \\ \sqrt{a} & y \end{vmatrix}^{13}$.

28. Find the sixth term of $(x\sqrt{y} + y\sqrt{x})^9$.

29. Find the middle terms of $(1 - a^{\frac{2}{5}})^7$.

30. Number the terms of the expansion of $(a + b)^6$, and with these numbers as abscissas and the coefficients of the corresponding terms as ordinates, plot points.

31. On the same coordinate paper on which points described in Problem 30 are plotted, graph points for the expansion of $(a + b)^{10}$.

32. Use the binomial theorem to find $(102)^5$.

Hint: $102 = (100 + 2)$.

Use the binomial theorem to find

33. $(99)^6$. **34.** $(51)^5$. **35.** $(.98)^6$.

36. $(1.1)^{10}$, correct to four significant figures.

37. $(1.1)^{15}$, correct to four significant figures.

CHAPTER X

VARIATION

61. Direct variation. In Chapter III we have seen that if y is a function of x, then in general y changes when x changes. We might say that y varies when x varies, but the word "varies" has come to have a more restricted meaning when used in this connection. Each of the statements "y varies as x," "y varies directly as x," "y is proportional to x," "y is directly proportional to x," means that the ratio of y to x is constant. That is,

$$\frac{y}{x} = k, \text{ or } y = kx.$$

The constant k is called the **constant of variation.** The expression "y varies as x" is often written

$$y \propto x.$$

The area of a circle varies as the square of its radius. That is, $A \propto r^2$, or $A = kr^2$, if A represents the area and r the radius. Here, k, the constant of variation, is equal to π. If a train moves with a uniform speed, the distance s traversed varies as the time t. That is, $s \propto t$, or $s = kt$.

62. Inverse variation. Each of the statements "y varies inversely as x," "y is inversely proportional to x" means that y is equal to the product of the reciprocal of x and a constant. That is,

$$y = \frac{k}{x}.$$

The volume of a gas varies inversely as the pressure. That is,

$$v = \frac{k}{p},$$

if v represents volume and p pressure.

63. Joint variation. The statement "z varies jointly as x and y" means that z equals the product of xy and a constant.

That is, $$z = kxy.$$

The distance which a train moving with a uniform speed traverses varies jointly as the speed and the time, or $d = kvt$, where d is the distance covered, v the speed, and t the time. In this case $k = 1$, if v and d are measured with the same unit of length.

64. Combined variation. The statement "z varies directly as x and inversely as y" means that z varies jointly as x and the reciprocal of y. That is,

$$z = \frac{kx}{y}.$$

The attraction F of any two masses m_1 and m_2 for each other varies as the product of the masses and inversely as the square of the distance r between the two bodies. That is,

$$F = \frac{km_1m_2}{r^2}.$$

EXERCISES AND PROBLEMS

Write the following statements, 1 to 8, in the form of equations.

1. y varies as x, and $y = 64$, when $x = 2$.

SOLUTION: $$y = kx.$$

Substituting 64 for y and 2 for x, gives

$$64 = 2k, \text{ or } k = 32.$$

Hence, $$y = 32x.$$

2. y is directly proportional to x, and $y = 18$ when $x = 6$.

3. s varies as t^2, and $s = 64$ when $t = 2$.

4. p varies inversely as v, and $v = 128$ when $p = 16$.

5. z varies jointly as x and y. When $x = 2$, and $y = 3$, it is found that $z = 120$.

6. z varies directly as x and inversely as y. When $x = 2$ and $y = 3$, it is found that $z = 120$.

7. The volume V of a sphere varies directly as the cube of its radius r.

8. The volume V of a circular cylinder varies jointly as its altitude h and the square of its radius r.

9. The number of feet a body falls varies directly as the square of the number of seconds occupied in falling. If the body falls 16.1 feet the first second, how many feet will it fall in 5 seconds? In 10 seconds?

10. The safe load of a horizontal beam supported at both ends varies jointly as the breadth and square of the depth, and inversely as the length between supports. If a 2×6 inches white pine joist, 10 feet long between supports, safely holds up 800 pounds, what is the safe load of a 4×8 beam of the same material 15 feet long?

11. The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the wind is 15 miles per hour, the pressure on a square foot is one pound. What is the velocity of the wind when the pressure on a square yard is 25 pounds?

12. The pressure of a gas in a tank varies jointly as its density and its absolute temperature. When the density is 1 and the temperature 300°, if the pressure is 15 pounds per square inch, what is the pressure when the density is 3 and the temperature 320°?

13. Write in the form of an equation the following physical law: the bend b of a rod supported at both ends varies directly as the weight m hung at its middle point, directly as the cube of the length l of the rod between supports, inversely as the width w of the rod, and inversely as the cube of the depth d.

14. A beam 15 feet long, 3 inches wide, and 6 inches deep when supported at each end can bear safely a maximum load of 1800 pounds. What is the greatest weight that can safely be placed on a beam of the same material 18 feet long, 4 inches wide, and 4 inches deep? (See Prob. 10.)

15. A plank 10 feet long, 10 inches wide, and 2 inches thick is supported at both ends. A weight of 180 pounds hung at the middle makes it bend 3 inches. How much will the plank bend if placed on edge? (See Prob. 13.)

16. The area of the top of a well-designed chimney varies as the quantity of coal used per hour and inversely as the square root of the height of the chimney. The top of a 150-foot chimney connected with a furnace using 11,000 pounds of coal per hour is 27 square feet. What should be the area of the top of a 100-foot chimney connected with a furnace using 2500 pounds of coal per hour?

17. If a heavier weight draws up a lighter one by means of a cord passed over a pulley, the number of feet passed over by each weight in a given time varies directly as the difference of the weights and inversely as their sum. If 5 pounds draw up 3 pounds 16.1 feet in 2 seconds, how high will 10 pounds draw 9 pounds in 2 seconds?

18. The time of a railway journey varies directly as the distance and inversely as the speed. The speed varies directly as the square root of the quantity of coal used per mile and inversely as the number of cars in the train. In a journey of 32 miles in $\frac{3}{4}$ hour with 12 cars, $\frac{1}{4}$ ton of coal is used. How much coal will be consumed in a journey of 64 miles in 2 hours with 10 cars?

19. A paper disk is placed midway between two sources of light which are 12 feet apart. If the amount of light falling on the disk varies inversely as the square of the distance from the source of light, show that if the disk is moved parallel to itself a distance $2\sqrt{3}$ feet, the whole amount of light falling on the disk is trebled.

20. How far must the disk in Problem 19 be moved from a point midway between the two lights so that the total amount of light on the disk is doubled?

21. A solid spherical mass of glass 2 inches in diameter is blown into a hollow spherical shell whose outer diameter is 4 inches. If the volume of a sphere varies as the cube of the diameter, what is the thickness of the shell?

22. Kepler's third law states that the square of the number of years it takes a planet to revolve about the sun varies directly as the cube of the distance of the planet from the sun. Let the distance from the earth to the sun be 1. How long would it take a planet whose distance from the sun is 100 to complete one revolution?

23. The number of inches a body falls in one second varies inversely as the square of the distance from the earth's center. At the surface of the earth a body falls 193 inches in one second. How far would it fall in one second if it were as far away as the moon? (The distance of the moon from the earth's center may be taken as sixty times the radius of the earth.)

24. The crushing load of a solid square oak pillar varies directly as the fourth power of its thickness and inversely as the square of its length. If a four-inch pillar 8 feet high is crushed by a weight of 196,320 pounds, what weight will crush a pillar of the same wood 6 inches thick and 12 feet high?

CHAPTER XI

PROGRESSIONS

65. Arithmetical progressions. An **arithmetical progression** is a sequence of numbers each of which differs from the next preceding one by a fixed number called the **common difference.** Thus,

$$2, 4, 6, 8, \cdots$$

is an arithmetical progression with the common difference 2. In the arithmetical progression

$$10, 8, 6, 4, 2, \cdots$$

the common difference is -2.

The numbers of the sequence are called the **terms** of the progression.

66. Elements of an arithmetical progression. Let a represent the first term, d the common difference, n the number of terms considered, l the nth, or last term, and s the sum of the sequence. The five numbers a, d, n, l, and s are called the *elements* of the arithmetical progression.

67. Relations among the elements. Since a is the first term, we have, by definition of an arithmetical progression,

$$\begin{aligned} a + d &= \text{second term}, \\ a + 2\,d &= \text{third term}, \\ a + 3\,d &= \text{fourth term}, \\ &\cdots \\ a + (n-1)d &= n\text{th term}. \end{aligned}$$

That is,

$$\boldsymbol{l = a + (n-1)d.} \qquad (1)$$

The sum of an arithmetical progression may be written in each of the following forms:

$$s = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l,$$
$$s = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$$

By addition

$$2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l)$$
$$= n(a+l).$$

Therefore,
$$s = \frac{n}{2}(a+l). \qquad (2)$$

Whenever any three of the five elements are given, equations (1) and (2) make it possible to find the remaining two elements.

Exercise. Establish formulas (1) and (2) by mathematical induction.

68. Arithmetical means. The first and last terms of an arithmetical progression are called the **extremes**, while the remaining terms are called the **arithmetical means.** To insert a given number of arithmetical means between two numbers it is only necessary to determine d by the use of equation (1) and to write down the terms by the repeated addition of d.

EXERCISES

Find l and s for the following seven sequences:

1. 2, 11, 20, ··· to 10 terms.

SOLUTION: Here,
$$l = a + (n-1)d.$$
$$a = 2,\ d = 9,\ n = 10.$$
$$l = 2 + 9 \cdot 9 = 83.$$
$$s = \frac{n}{2}(a+l)$$
$$= 5(2+83) = 425.$$

2. $-2, -5, -8,$ ··· to 20 terms.

3. 3, 7, 11, ··· to 15 terms.

4. 5, 1, -3, to 20 terms.

5. $\frac{2}{3}, \frac{7}{12}, \frac{1}{2}$, to 8 terms.

6. $7, \frac{29}{4}, \frac{15}{2}$, ··· to 16 terms.

7. $-\frac{7}{3}, -\frac{40}{21}, -\frac{31}{21}$, ··· to 17 terms.

8. Given $a = 19,\ d = -2,\ s = 91$; find n and l.

9. Given $d = 4,\ n = 15,\ l = 59$; find a and s.

10. Given $l = -\frac{1}{2},\ n = 14,\ d = \frac{1}{2}$; find a and s.

11. Given $a = 12,\ l = -76,\ s = -520$; find n and d.

12. Given $a = 7,\ l = \frac{43}{4},\ d = \frac{1}{4}$; find n and s.

13. Given $a = -4$, $l = -64$, $n = 21$; find d and s.

14. Given $d = \frac{1}{2}$, $l = \frac{7}{7}\frac{4}{4}$, $s = 1\frac{6}{4}\frac{3}{3}$; find a and n.

15. Given $l = 50$, $n = 50$, $s = 1275$; find a and d.

16. Given $d = 10$, $n = 10$, $s = 10$; find a and l.

17. Given $a = 7$, $n = 7$, $s = 7$; find d and l.

18. Insert 6 arithmetical means between 3 and 8.

SOLUTION: We have to find d, when $a = 3$, $l = 8$, and $n = 6 + 2 = 8$.

Since $$l = a + (n-1)d,$$

we have $$8 = 3 + 7d, \text{ or } d = \tfrac{5}{7}.$$

Hence, the 6 arithmetical means between 3 and 8 are

$$\tfrac{26}{7}, \tfrac{31}{7}, \tfrac{36}{7}, \tfrac{41}{7}, \tfrac{46}{7}, \tfrac{51}{7}.$$

19. Insert 3 arithmetical means between 2 and 14.

20. Find the arithmetical mean between $10\frac{2}{3}$ and 16.

21. Insert 9 arithmetical means between 1 and -1.

69. Geometrical progressions. A **geometrical progression** is a sequence of numbers in which the same quotient is obtained by dividing any term by the preceding term. This quotient is called the **ratio.** Thus,

$$3, 6, 12, 24, \cdots$$

is a geometrical progression with a ratio 2.

70. Elements of a geometrical progression. The elements are the same as those for an arithmetical progression with one exception. Instead of the common difference of an arithmetical progression, we have here a ratio represented by r.

71. Relations among the elements. If a represents the first term, then

$$\begin{aligned} ar &= \text{second term},\\ ar^2 &= \text{third term},\\ ar^3 &= \text{fourth term},\\ &\cdots\\ ar^{n-1} &= n\text{th term}. \end{aligned}$$

That is, $$l = ar^{n-1}. \tag{1}$$

By definition,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \tag{2}$$

Then, $$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{3}$$

Subtracting members of (2) from members of (3), we have

$$sr - s = ar^n - a.$$

Hence, $$s = \frac{ar^n - a}{r - 1} = a\frac{(1 - r^n)}{1 - r}. \qquad (4)$$

Since $l = ar^{n-1}$, (4) may be written in the form

$$s = \frac{rl - a}{r - 1}. \qquad (5)$$

Here, as in an arithmetical progression, whenever any three of the five elements are given, relations (1) and (5) make it possible to find the other two.

Exercise. Establish formulas (1) and (4) by mathematical induction.

72. Geometrical means. The first and last terms of a geometrical progression are called the **extremes**, while the remaining terms are called the **geometrical means.** To insert n geometrical means between two given numbers is to find a geometrical progression of $n+2$ terms having the two given numbers for extremes.

EXERCISES

1. Given $a = 1$, $r = 3$, $n = 9$; find l and s.

2. Given $a = -2$, $r = 3$, $n = 8$; find l and s.

3. Given $a = \frac{1}{2}$, $r = \frac{1}{2}$, $n = 10$; find l and s.

4. Given $a = 6$, $r = -2$, $n = 8$; find l and s.

5. Given $s = 242$, $a = 2$, $n = 5$; find r and l.

6. The 3d term of a geometrical progression is 3, and the 6th term is 81. What is the 10th term?

7. What is the 7th term of a geometrical progression whose first term is 2 and third term 3?

8. What is the sum of the first 5 terms of a geometrical progression whose first term is 2 and third term 8?

9. The first term of a geometrical progression is 3, and the last term 81. If there are four terms in the geometrical progression, find the common ratio and the sum of the series.

10. Insert one geometrical mean between 7 and 343.

11. Insert two geometrical means between 2 and 1024.

12. Insert four geometrical means between 12 and $\frac{3}{8}$.

13. What is the eighth term of the progression a^2, ab, b^2, ...?

14. If each term of a geometrical progression is multiplied by the same number, show that the products form a geometrical progression.

73. Number of terms infinite. Consider the geometrical progression

$$\tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \cdots.$$

It may at first thought appear that the sum of the first n terms of this progression could be made to exceed any finite number previously assigned by making n large enough. That this is not the case and that the sum can never exceed unity, will be seen from the following illustration. Conceive a particle moving in a straight line towards a point one unit distant in such a way as to describe $\frac{1}{2}$ the distance in the 1st second, $\frac{1}{2}$ the remaining distance in the 2d second, $\frac{1}{2}$ the remaining distance in the 3d second, and so on indefinitely. This is represented in Fig. 17.

The distance AB represents one unit of distance. In the first second the particle moves from A to P_1. In the 2d second it moves from P_1 to P_2, and so on. The total distance traversed by the particle in n seconds is given by the sum

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \text{ to } n \text{ terms},$$

which sum cannot exceed nor equal 1, no matter how many terms we take, but can be made to differ from 1 by as small a positive number as we please by making the number of terms large enough.

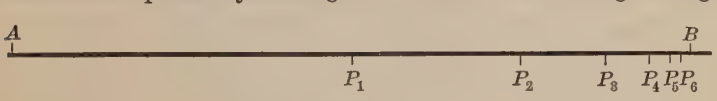

FIG. 17.

Thus, when $n = 10$, the sum is $\frac{1023}{1024}$ (Prob. 3, Art. 72). In this illustration, 1 is said to be the limiting * value of the sum of the first n terms of the series. If s_n represents the sum of the first n terms of the series, we write

$$s = \lim_{n \to \infty} s_n = 1,$$

which reads, "the limit of s_n as n increases **beyond bound** is 1." †

* For definition of limit, see Art. 150.

† The symbol "$n \to \infty$" stands for "n increases beyond bound," or its equivalent "n becomes infinite."

The limit s is called the **sum** of the geometrical progression with infinitely many terms.

For any geometrical progression in which the ratio is less than 1, the above argument can be repeated, and it can be shown that there is a limiting value to the sum of the first n terms of such a series. In Art. 71, we have shown that the sum of the geometrical progression

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

is given by $$s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

We may then write

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r}. \quad \text{(See Art. 152.)}$$

It will be proved in the chapter on Limits (Art. 154) that

$$\lim_{n\to\infty} \frac{ar^n}{1-r} = 0 \text{ when } |r| < 1.$$

Hence, $$s = \lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

74. Series. A **series** is an expression which consists of the sum of a sequence of terms. Thus, the indicated sum of the terms of a progression is often called a series.

A **finite** series is one which has a limited number of terms.

An **infinite** series is one in which the number of terms is infinite; that is, the number of terms has no bound.

75. Repeating decimals. Repeating decimals furnish good illustrations of infinite series which are at the same time the sum of the terms of a geometrical progression with infinitely many terms. For example, .33333 ··· may be written as the series

$$.3 + .03 + .003 + .0003 + \cdots,$$

where $a = .3$ and $r = .1$. The limit of the sum of n terms of this series as the number n increases indefinitely is $\frac{1}{3}$. Again, .9828282 ··· may be written

$$.9 + .082 + .00082 + \cdots$$

where the terms after the first form a geometrical progression in which $a = .082$ and $r = .01$.

The expression "limit of the sum of n terms of the infinite series as n increases beyond bound" is often abbreviated by saying merely "**sum of the infinite series.**"

EXERCISES

Find the sum of the following infinite series:

1. $\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$.
2. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.
3. $\frac{2}{3} - \frac{1}{3} + \frac{1}{6} - \cdots$.
4. $24 - 12 + 6 - 3 + \cdots$.
5. $4 - 3 + \frac{9}{4} - \cdots$.
6. $-4 - \frac{1}{2} - \frac{1}{16} - \cdots$.

Find the limiting value of the following repeating decimals:

7. .636363 ....
8. .44444 ....
9. .83333 ....
10. .363636 ....
11. 40.909090 ....
12. .54123123 ....

76. Harmonical progressions. Three or more numbers are said to form a **harmonical progression** if their reciprocals form an arithmetical progression. The term "harmonical" as here used comes from a property of musical sounds. If a set of strings of uniform tension whose lengths are proportional to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ be sounded together, the effect is harmonious to the ear. The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$$

is a harmonical progression since the reciprocals form the arithmetical progression

$$1, 2, 3, 4, 5, \cdots.$$

77. Harmonical means. To find n harmonical means between two numbers, find n arithmetical means between the reciprocals of these numbers. The reciprocals of the arithmetical means are the harmonical means.

EXERCISES

1. Insert three harmonical means between 3 and 12.
2. Insert two harmonical means between 2 and $\frac{1}{8}$.
3. Insert four harmonical means between $\frac{1}{12}$ and $\frac{1}{2}$.
4. What is the harmonical mean between a and b?
5. Show that 4, 6, 12 are in harmonical progression, and continue the series for two terms in each direction.

6. If A, G, and H stand respectively for the arithmetical, geometrical, and harmonical means between two numbers a and b, show that $G^2 = AH$.

7. The numbers a, b, c are in arithmetical progression, and b, c, d are in harmonical progression; prove that $ad = bc$.

PROBLEMS

1. What distance will an elastic ball traverse before coming to rest if it be dropped from a height of 20 feet and if after each fall it rebounds one third of the height from which it falls?

2. If a falling body descends $16\frac{1}{2}$ feet the first second, 3 times this distance the next, 5 times the next, and so on, how far will it fall the 30th second, and how far altogether in 30 seconds?

3. Assume that a baseball will fall 16 feet the first second, 48 the next, 80 the next, and so on. A baseball was dropped from the top of Washington Monument, 550 feet high, and caught by an American League catcher. About how fast was the ball falling when caught?

4. A swinging pendulum is brought gradually to rest by friction of the air. If the length of the first swing of the pendulum bob is 30 centimeters, and the length of each succeeding swing is $\frac{1}{10}$ less than the preceding one, what is the distance passed over in the fifth swing?

5. What is the total distance passed over by the pendulum bob described in Problem 4 in 5 swings?

6. A ball rolls down an incline of 20 degrees, 5.47 feet the first second, and in each succeeding second, 10.94 feet more than in the preceding second. How far will it roll in 10 seconds?

7. Two straight lines l_1 and l_2 meet at the point 0. From a point in l_1 drop a perpendicular to l_2. From the foot of this perpendicular drop a perpendicular to l_1 and so on indefinitely. If the lengths of the first and second perpendiculars are 6 and 5 respectively, what is the sum of the lengths of all the perpendiculars?

8. In a raffle, tickets marked 1, 2, 3, 4, etc., are shaken up in a hat and drawn by the purchasers one at a time. The price of a ticket is the number of cents corresponding to the number on the ticket. If the raffled article is worth \$10, what is the least number of tickets which will insure no loss to the vendors of the tickets?

9. A person contributes one cent and sends letters to three friends asking each to contribute one cent to a certain charity and to write a similar letter to each of three friends, each of whom is to write three letters, — and so on until ten sets of letters have been written. If all respond, how much money will the charity receive?

10. Twenty-five stones are placed in a straight line on the ground at intervals of 4 feet. 10 feet from the end of the row is a basket. A runner starts from the basket and picks up the stones and carries them, one at a time, to the basket. How far does he run altogether?

11. An employer hires a clerk for five years at a beginning salary of $500 per year with either a rise of $100 each year after the first, or a rise of $25 every six months after the first half year. Which is the better proposition for the clerk?

12. The population of a town is 10,000. It loses annually 2 per cent of its population by deaths and gains 3 per cent by births, and at the end of each year, it has gained 100 people as a result of movings into and away from the town. What will the population be in 10 years?

13. Find the limit of the sum of the series

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\cdots \text{ where } n>0.$$

14. What is the sum of the first n odd numbers?

15. What is the equation whose roots are the arithmetical and the harmonical means between the roots of $x^2-16x+48=0$?

16. If $\frac{1}{y-x}, \frac{1}{2y}, \frac{1}{y-z}$ form an arithmetical progression, show that x, y, and z form a geometrical progression.

17. What is the number which added to each of the numbers 1, 3, 2, produces a geometrical progression?

18. The fourth term of a geometrical progression is 108, the seventh term is 864. What is the 10th term?

19. There are four numbers in arithmetical progression. When these numbers are increased by 2, 4, 8, 15 respectively, the sums are in geometrical progression; find the four numbers in arithmetical progression.

20. Three numbers whose sum is 18 are in arithmetical progression. If you multiply the first by 2, the second by 3, and the third by 6, the resulting products form a geometrical progression. Find the three numbers.

21. Find the present value of ten annual payments of $1000 each made at the ends of the next ten years, if money is worth 5 per cent compounded annually.

CHAPTER XII

COMPLEX NUMBERS

78. Number systems. If our number system consisted of zero and positive integers only, the solution of an equation such as $3x - 2 = 0$ would be impossible; for no number in the system considered satisfies this equation. We can extend the number system so as to include the class of numbers to which the solution belongs. These new numbers are the *rational fractions.*

While the solution of $3x - 2 = 0$ is possible in a number system composed of zero, positive integers, and rational fractions, the solution of an equation such as $x + 4 = 0$ is impossible. To meet the demands of such equations, we find it expedient again to extend the number system so as to include the negative numbers. In a number system thus extended an equation $ax + b = 0$, where a and b are any integers or fractions, has a solution.

The solution of an equation such as $x^2 = 2$ demands a further extension of our number system. It must be made to include irrational numbers, that is, numbers which cannot be represented by the quotient of two integers (see p. 23). But the number system thus extended is not sufficient to meet all the demands of the equations met in algebra. In this number system it is impossible to solve certain quadratic equations, for example, the equations $x^2 + 1 = 0$ and $x^2 - 6x + 13 = 0$. It is necessary again to extend the system so as to include numbers of the form $a + bi$, where a and b are real numbers, discussed in Art. 1; and where i is a symbol whose square is -1, that is, $i = \sqrt{-1}$ (see Art. 22). These numbers are usually called **complex numbers** and sometimes imaginary numbers. When $a = 0$ they are called **pure imaginary** numbers.

The term "imaginary number" is here used in a technical sense. The numbers are imaginary in the same sense that a

fraction, a negative number, or an irrational number is imaginary for a number system consisting of positive integers.

At this point the question may be asked, — In working with this new system which includes complex numbers, may we not find it necessary to add new numbers, at present unknown, just as we found it necessary to add fractions, negative numbers, and irrational numbers to our system of positive integers? The answer to this question is that the system of complex numbers is sufficient to meet the demands of the equation.

While we have seen that the solution of equations with integral coefficients demands fractions, negative numbers, irrational numbers, and complex numbers, it is not to be inferred that all numbers are roots of equations with integral or rational coefficients. For example, the irrational number π cannot be the root of an equation with rational coefficients. The proof is beyond the scope of this book.*

79. Graphical representation of complex numbers. We have seen that all real numbers may be represented by points on a straight line. The complex number $x + iy$ depends on two real numbers x and y, and may be represented graphically by a point in a plane. Two lines, $X'X$, $Y'Y$, are drawn perpendicular to each other and intersecting at O, Fig. 18. To represent the number $2 + 3i$, measure off on $X'X$ to the right the distance 2, and up the distance 3. In general, the graph of the number $x + iy$ is the point whose coordinates are (x, y). The line $X'X$ is often called the axis of "reals" and the line $Y'Y$ the axis of imaginaries.

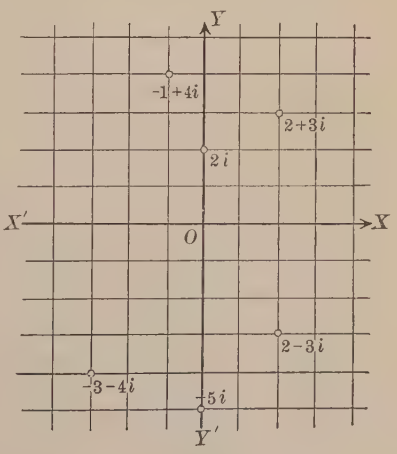

FIG. 18.

* See Klein, *Famous Problems in Elementary Geometry*, translation by Beman and Smith, p. 68.

*It is often convenient to represent complex numbers by another method. Connect the point which represents $x + iy$ with the origin as in Fig. 19. Let the length of this line be r. The point can then be represented by giving the length r and the angle θ. From the figure

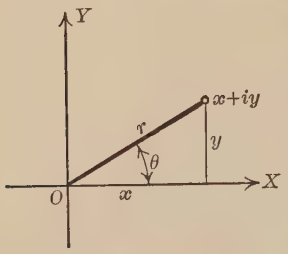

FIG. 19

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$x^2 + y^2 = r^2.$$

Hence, the number $x + iy$ may be written in the form

$$x + iy = r(\cos\theta + i\sin\theta).$$

This form is called the **polar** form of a complex number. The angle θ is called the **argument** or **amplitude**, the length r the **modulus** or **absolute value** of the complex number. It should be noted that the complex numbers include all real numbers. In Fig. 18, the real numbers are represented by points on the line $X'X$. The pure imaginary numbers are represented by points on the line $Y'Y$.

80. Equal complex numbers. If two complex numbers $a + bi$ and $c + di$ are equal, then $a = c$ and $b = d$. For, if

$$a + bi = c + di, \tag{1}$$

by transposing,

$$a - c = (d - b)i. \tag{2}$$

Unless $a - c = d - b = 0$, we should have $a - c$, a real number, equal to $(d - b)i$, an imaginary number.

Conversely, if $a = c$, and $b = d$,

$$a + bi = c + di.$$

Hence, *when any two expressions containing imaginary and real terms are equal to each other, we may equate the real parts and the imaginary parts separately.*

In particular, if $a + bi = 0$, $a = 0$ and $b = 0$.

*The remainder of this article and the articles marked * may be omitted by those who have not studied trigonometry.

EXERCISES

Represent graphically the following numbers and in each case find the argument and the modulus.

1. $2-3i$.

SOLUTION: The number is represented in Fig. 18.
The modulus r is given by

$$r=\sqrt{x^2+y^2}=\sqrt{4+9}=\sqrt{13}.$$

To find the argument θ we have

$$\tan\theta=\frac{y}{x}=-\frac{3}{2},\quad \theta=\arctan-\frac{3}{2};\quad \sin\theta=-\frac{3}{\sqrt{13}}.$$

2. $-2-3i$. **3.** $-2+3i$. **4.** $1+i$.

5. $1-i$. **6.** $5-4i$. **7.** $-i$.

8. $4i$. **9.** $-6i$. **10.** $\frac{5}{2}i$.

11. $4+0i$. **12.** -5. **13.** $\frac{3}{3}-\frac{i}{2}$.

14. $0.7+1.1i$.

Write the following complex numbers in the form $x+iy$.

15. $3(\cos 30^\circ+i\sin 30^\circ)$. **16.** $2(\cos 60^\circ+i\sin 60^\circ)$.

17. $2(\cos 150^\circ+i\sin 150^\circ)$. **18.** $4(\cos 90^\circ+i\sin 90^\circ)$.

19. $(\cos 225^\circ+i\sin 225^\circ)$. **20.** $6(\cos 0^\circ+i\sin 0^\circ)$.

21. $\frac{3}{2}(\cos 270^\circ+i\sin 270^\circ)$. **22.** If $(x+1)+i(y-1)=0$, what are the values of x and y?

What must be the value of x and y in order that the following equations may be true?

23. $x+y+i(x-y)=2+4i$. **24.** $2x+7y+i(3x-2y)=-3-i$.

25. $2x^2+y^2+i(x-y)=1+i$. **26.** $3x+xyi+2y-ix-3y-5=0$.

27. $x^2+ix+iy+y^2=130+8i(x-y)$.

81. Addition and subtraction of complex numbers. We assume that the number i like other numbers obeys all the laws of algebra. Given two complex numbers $a+bi$, $c+di$, we may write the sum and difference.

Thus,
$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

Hence, to add (or subtract) complex numbers, add (or subtract) the real and imaginary parts separately. The result is a complex number.

To add two complex numbers, $a + bi$ and $c + di$, graphically, we represent the numbers as points A and B in Fig. 20. Connect each point with the origin O. Complete the parallelogram, having OA and OB for adjacent sides. The vertex P represents the sum of the two given complex numbers. For, from the figure the coordinates of the fourth vertex are OQ and QP. But

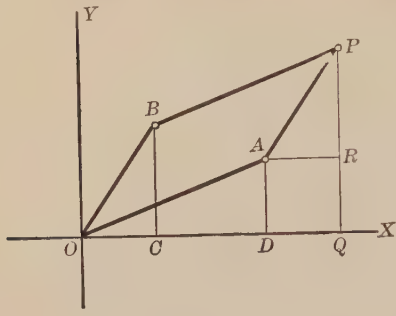

Fig. 20.

$$OQ = OD + DQ = a + c,$$
$$QP = QR + RP = b + d.$$

Hence, P represents the point $(a + c) + (b + d)i$ which is the sum of $a + bi$ and $c + di$.

To subtract one complex number from another graphically, say $c + di$ from $a + bi$, we graph the points which represent $-c - di$ and $a + bi$, and proceed as for addition.

EXERCISES

Perform the following operations algebraically and graphically.

1. $(1 + i) + (2 + 3i)$.
2. $(2 + 2i) + (1 - 3i)$.
3. $(2 + i) - (1 + 4i)$.
4. $(3 - 5i) + (3 + 5i)$.
5. $(3 - 5i) - (3 + 5i)$.
6. $(-3 - 4i) + (5 - i)$.
7. $(0 + 3i) + (1 - 4i)$.
8. $(6 + 0i) + (-3 + 7i) - (4 + 2i)$.
9. $(4 + i) - (3 + 3i) + (1 - i)$.
10. $(4 + 3i) - (4 + 3i)$.

* **82. Multiplication of complex numbers.** Let $a + ib$ and $c - id$ be any two complex numbers. Since i obeys all the laws of algebra, we have

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad).$$

The result is a complex number. To multiply two complex numbers graphically, let the two numbers $a + ib$, $c + id$ be represented by the points P_1 and P_2 (Fig. 21). Reducing to the polar form, we have

$$a + bi = r_1(\cos\theta_1 + i\sin\theta_1),$$
$$c + di = r_2(\cos\theta_2 + i\sin\theta_2).$$

By actual multiplication,

$$(a + bi)(c + di)$$
$$= r_1 r_2[\cos\theta_1 \cos\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) - \sin\theta_1 \sin\theta_2]$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

Hence, the modulus of the product of two complex numbers is the product of their moduli and the argument is the sum of their arguments.

The point P, which represents $(a + bi)(c + di)$ may then be constructed by drawing through O a line making an angle $\theta = \theta_1 + \theta_2$ with the line OX, and constructing on this length a segment OP, whose length is $r_1 r_2$.

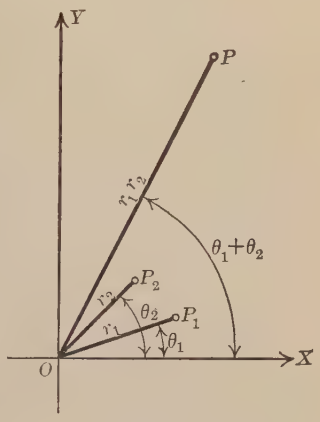

FIG. 21.

83. Conjugate complex numbers. Numbers which differ only in the sign of the imaginary parts are called **conjugate numbers**. Thus, $3+2i$ and $3-2i$ are conjugate. Since

$$(a + bi) + (a - bi) = 2a,$$
$$(a + bi)(a - bi) = a^2 + b^2,$$

and
$$(a + bi) - (a - bi) = 2bi,$$

we see that the sum and the product of two conjugate complex numbers are real numbers, but the difference of two conjugate complex numbers is an imaginary number.

EXERCISES

Multiply both analytically and graphically, finding the arguments and moduli of the products.

1. $(3 + \sqrt{3}\,i)(2 + 2i)$.

SOLUTION.

$(3 + \sqrt{3}\,i)(2 + 2i) = 6 + 6i + 2\sqrt{3}\,i + 2\sqrt{3}\,i^2 = 6 - 2\sqrt{3} + i(6 + 2\sqrt{3})$.

Putting the numbers in the polar form, we have,

$$3 + \sqrt{3}\,i = 2\sqrt{3}(\cos 30° + i \sin 30°),$$
$$2 + 2i = 2\sqrt{2}\,(\cos 45° + i \sin 45°).$$

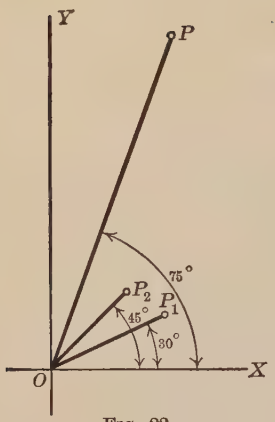

FIG. 22.

Hence,

$r_1 = 2\sqrt{3}$, $r_2 = 2\sqrt{2}$, $\theta_1 = 30°$, $\theta_2 = 45°$.

The modulus of the product is, then,

$$r_1 r_2 = 4\sqrt{6},$$

and the argument is 75°.

Let P_1 and P_2 in Fig. 22 represent the two given numbers. Through O draw a line making an angle of 75° with the line OX. On this line measure off the distance

$$OP = 4\sqrt{6}.$$

The point P then represents the product of the two numbers.

2. $(3 + \sqrt{3}\,i)(2 + \frac{2}{3}\sqrt{3}\,i)$.

3. $(1 + \sqrt{3}\,i)(4 + \frac{4}{3}\sqrt{3}\,i)$.

4. $(\sqrt{2} + \sqrt{-2})^2$.

5. $(1 + i)^4$.

6. $(1 + i)^2(2 - 2\sqrt{3}\,i)$.

7. $(-2 + 2i)(2 + 2i)$.

8. $(-2 - 2i)(2 + 2i)$.

9. $(1 + i)(1 + 2i)(1 + 3i)$.

10. $(0 + 3i)(2 + i)$.

11. $(0 + 2i)(0 - 2i)$.

* **84. De Moivre's theorem.** If two complex numbers are equal, then as a special case of Art. 82, we have

$$r(\cos\theta + i\sin\theta)\,r(\cos\theta + i\sin\theta) = r^2(\cos\theta + i\sin\theta)^2$$
$$= r^2(\cos 2\theta + i\sin 2\theta).$$

Multiplying both sides of this identity by $r(\cos\theta + i\sin\theta)$, we have

$$r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta),$$

and it can easily be proved by mathematical induction that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta),$$

where n is any positive integer.

This relation is known as **De Moivre's theorem,** and holds also for fractional values of the exponent.

To prove the theorem when the exponent is the reciprocal of a positive integer, consider the expression $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$ in which n is a positive integer.

Let $\theta = n\phi,$

then $(\cos\theta + i\sin\theta)^{\frac{1}{n}} = (\cos n\phi + i\sin n\phi)^{\frac{1}{n}}$

$$= [(\cos\phi + i\sin\phi)^n]^{\frac{1}{n}} = \cos\phi + i\sin\phi$$

$$= \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}.$$

De Moivre's theorem thus gives an easy method of finding any power or any root of a complex number.

The proof can be readily extended to the case of an exponent which is any rational fraction.

* **85. Roots of complex numbers.** From Art. 84, the nth root of $x + iy$ is

$$(x+iy)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).$$

If m be any integer, $\cos(\theta + m\,360°) = \cos\theta,$

$$\sin(\theta + m\,360°) = \sin\theta.$$

We may then write

$$(x+iy)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = [r\{\cos(\theta + m\,360°) + i\sin(\theta + m\,360°)\}]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\frac{\theta + m\,360°}{n} + i\sin\frac{\theta + m\,360°}{n}\right].$$

If now we let m take the values $0, 1, 2, 3, \cdots, n-1$, we find n results, all different numbers whose nth powers are $x + iy$. We may then state the following

THEOREM. *Any number has n distinct nth roots.*

EXERCISES

Using De Moivre's theorem, find the indicated powers and roots.

1. $(3 + \sqrt{3}\,i)^4$.

SOLUTION: Writing $3 + \sqrt{3}\,i$ in the polar form,

$$3 + \sqrt{3}\,i = 2\sqrt{3}(\cos 30° + i\sin 30°).$$

By De Moivre's theorem,

$$(3 + \sqrt{3}\,i)^4 = [2\sqrt{3}(\cos 30° + i\sin 30°)]^4$$

$$= 144(\cos 120° + i\sin 120°)$$

$$= 144(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i)$$

$$= -72 + 72\sqrt{3}\,i.$$

2. $(3+\sqrt{3}\,i)^5$. **3.** $(3+\sqrt{3}\,i)^3$.

4. $(-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i)^3$. **5.** $(\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i)^6$.

6. $[2(\cos 10^\circ + i \sin 10^\circ)]^9$. **7.** $(2+2\,i)^5$. **8.** $(1+i)^{10}$.

9. $\sqrt[3]{-2+2\,i}$.

SOLUTION: Writing $-2+2\,i$ in the polar form, we have

$$-2+2\,i = 2\sqrt{2}\,(\cos 135^\circ + i \sin 135^\circ).$$

By De Moivre's theorem,

$$\sqrt[3]{-2+2\,i} = (-2+2\,i)^{\frac{1}{3}} = [2\sqrt{2}\,\{\cos(135^\circ + m\,360^\circ) + i \sin(135^\circ + m\,360^\circ)\}]^{\frac{1}{3}}$$
$$= \sqrt{2}\,[\cos(45^\circ + m\,120^\circ) + i \sin(45^\circ + m\,120^\circ)].$$

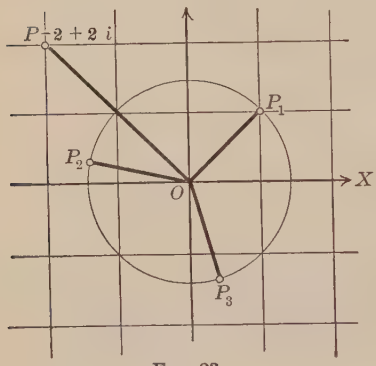

FIG. 23.

For $m = 0$, 1, and 2, this expression reduces to

$1+i$, $\sqrt{2}(\cos 165^\circ + i \sin 165^\circ)$,

and $\sqrt{2}\,(\cos 285^\circ + i \sin 285^\circ)$

respectively. Any one of these three numbers is a cube root of $-2+2\,i$. The points P_1, P_2, P_3 representing these three numbers lie at equal intervals on a circle of radius $\sqrt{2}$ (Fig. 23).

10. $\sqrt[3]{2+2\,i}$.

11. $\sqrt{-8+8\sqrt{3}\,i}$.

12. $\sqrt{3+\sqrt{3}\,i}$. **13.** $\sqrt{64(\cos 60^\circ + i \sin 60^\circ)}$.

14. $\sqrt[3]{64(\cos 60 + i \sin 60^\circ)}$. **15.** $\sqrt[6]{64(\cos 60 + i \sin 60^\circ)}$.

16. $\sqrt[10]{\cos 300^\circ + i \sin 300^\circ}$. **17.** $\sqrt[5]{-4-4\,i}$.

18. $\sqrt[5]{i}$.

HINT: Write in the form $\sqrt[5]{0+i}$.

19. $\sqrt[3]{27\,i}$. **20.** $\sqrt{4}$. **21.** $\sqrt[3]{1}$. **22.** $\sqrt[3]{8}$. **23.** $\sqrt[4]{81\,i}$.

Find all the roots of the following equations and represent them graphically.

24. $x^3 - 27 = 0$.

HINT: $x^3 = 27$. The roots of the equation are then the three cube roots of $27 + 0\,i$.

25. $x^5 - 1 = 0$. **26.** $x^5 - 32 = 0$. **27.** $x^3 - 1 = 0$.

28. $x^4 - 16 = 0$. **29.** $x^6 - 1 = 0$. **30.** $x^8 - 1 = 0$.

*** 86. Division of complex numbers.** The quotient of two complex numbers may be obtained as follows.

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\cdot\frac{c-id}{c-id}$$

$$= \frac{ac+bd-i(ad-bc)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{ad-bc}{c^2+d^2}.$$

This is a complex number. Writing the two given complex numbers in the polar form, we have

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2^2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1r_2[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}{r_2^2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

Hence, *the modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument is the difference of their arguments.*

If, in Fig. 24, P_1 and P_2 represent the points $a+ib$ and $c+id$ respectively, the point P which represents the quotient $\frac{a+ib}{c+id}$ may be constructed by drawing through O a line making an angle $\theta=\theta_1-\theta_2$ with the line OX, and constructing on this line a segment OP, whose length is $\frac{r_1}{r_2}$.

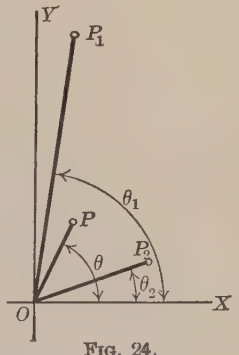

FIG. 24.

EXERCISES

Find the quotients of the following pairs of numbers, analytically and graphically.

1. $(4+4i)\div(2+\frac{2}{3}\sqrt{3}\,i)$.

Solution: $$\frac{4+4i}{2+\frac{2}{3}\sqrt{3}\,i}=\frac{4+4i}{2+\frac{2}{3}\sqrt{3}\,i}\cdot\frac{2-\frac{2}{3}\sqrt{3}\,i}{2-\frac{2}{3}\sqrt{3}\,i}=\frac{3+\sqrt{3}}{2}+\frac{3-\sqrt{3}}{2}i.$$

Writing both numbers in the polar form, we obtain

$$4+4i=4\sqrt{2}(\cos 45^\circ+i\sin 45^\circ),$$
$$2+\tfrac{2}{3}\sqrt{3}\,i=\tfrac{4}{3}\sqrt{3}(\cos 30^\circ+i\sin 30^\circ).$$

Hence, $r_1=4\sqrt{2}$, $r_2=\frac{4}{3}\sqrt{3}$, $\theta_1=45^\circ$, $\theta_2=30^\circ$.

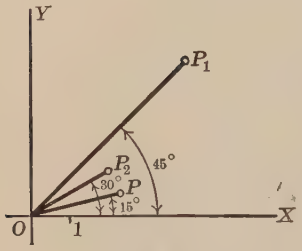

Fig. 25.

The modulus of the quotient is then $\frac{r_1}{r^2}=\sqrt{6}$, and the argument is $\theta=\theta_1-\theta_2=15^\circ$. Let P_1 and P_2 in Fig. 25 represent the two given numbers. Through O draw a line, making an angle of 15° with the line OX. Measure off on this line the distance $OP=\sqrt{6}$. The point P then represents the quotient of the two numbers.

2. $(-2+2\sqrt{3}\,i)\div(1+\sqrt{3}\,i)$.

3. $(-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i)\div(-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i)$.

4. $(4+4i)\div(1-i)$.

5. $(1+i)^2\div(2-2\sqrt{3}\,i)$.

6. $-4i\div(-2+2i)$.

7. $2\div(1+i)$.

8. $(-2+2i)\div(-4i)$.

9. $(1+i)\div(1-i)$.

10. $(3+\sqrt{3}\,i)\div(\sqrt{3}+3i)$.

Graph the following complex numbers and their reciprocals.

11. $1+i$. **12.** $3+\sqrt{3}\,i$. **13.** $2-2i$. **14.** $3+7i$. **15.** $-3+5i$.

CHAPTER XIII

THEORY OF EQUATIONS

87. The polynomial of the nth degree. The general form (Art. 27) of a polynomial of the nth degree is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

where n is a positive integer, $a_0, a_1, \cdots, a_n$ are independent of x, and $a_0 \neq 0$. The polynomial of the second degree has been discussed in connection with the quadratic equation.

When $f(x)$ is used in this chapter, it is to be understood to mean a polynomial in x.

EXERCISE

By comparing the following polynomials with the general form, determine n, a_0, a_1, $\cdots$, a_n.

(1) $f(x) = \frac{3}{5}x^5 + \frac{2}{7}x^4 + 8.$

(2) $f(x) = \frac{4}{7}x^8 + \frac{2}{5}x^4 + 10x^2 + ix^2.$

(3) $f(x) = (i + \sqrt{3})x^5 + 5x^2 + 10.$

88. Remainder theorem. *If $f(x)$ is divided by $x - r$, the remainder is $f(r)$.*

Given $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \tag{1}$$

Then, $$f(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n, \tag{2}$$

and $$f(x) - f(r) = a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + a_{n-1}(x - r). \tag{3}$$

But since $x - r$ is a factor of each of the expressions $x^n - r^n$, $x^{n-1} - r^{n-1}$, $\cdots$, $x - r$ (Ex. 11, Art. 58, p. 92), it is a factor of the right-hand member of (3).

If we should inclose the whole right-hand member of (3) in a parenthesis, we could remove the factor $x - r$, and call what remains inside the parenthesis $Q(x)$.

Then we have from (3),

$$f(x) - f(r) = (x - r)Q(x). \tag{4}$$

Transposing $f(r)$ and dividing by $x - r$, we have

$$\frac{f(x)}{x-r} = Q(x) + \frac{f(r)}{x-r},$$

which is what was to be proved.

COROLLARY. *If $f(x)$ vanishes when $x = r$, then $f(x)$ is exactly divisible by $x - r$.*

EXERCISES

1. Show, by the remainder theorem, that $x^n - a^n$ is exactly divisible by $x + a$ if n is even.

2. Show that $x^n + a^n$ is divisible by $x + a$ if n is odd.

3. Without performing the division, find the remainder when $x^3 - 3x^2 + 2x - 1$ is divided by $x - 2$.

4. By means of the remainder theorem, find a value for k such that $x^3 + 3kx^2 + 6x + 2$ is divisible by $x - 1$.

89. Synthetic division. The operation of dividing a polynomial by a binomial $x - r$ can be performed rapidly by means of a process called synthetic division. This rapid division, combined with the remainder theorem, gives a convenient method of evaluating $f(x)$ for different values of x.

For example, divide

$$5x^4 - 6x^3 + 8x^2 - 24x - 6 \text{ by } x - 2.$$

By the ordinary method

$$\begin{array}{r|l}
5x^4 - 6x^3 + 8x^2 - 24x - 6 & x - 2 \\
\underline{5x^4 - 10x^3} & \overline{5x^3 + 4x^2 + 16x + 8} \\
4x^3 + 8x^2 & \\
\underline{4x^3 - 8x^2} & \\
16x^2 - 24x & \\
\underline{16x^2 - 32x} & \\
8x - 6 & \\
\underline{8x - 16} & \\
+ 10 &
\end{array}$$

Manifestly, the work can be abridged by writing only the coefficients, thus,

$$\begin{array}{rrrrrr|l} 5- & 6+8- & 24- & 6 & \underline{1-2} \\ \underline{5-10} & & & & 5+4+16+8 \\ \quad +4+8 & & & & \\ \quad \underline{+4-8} & & & & \\ \quad\quad +16- & 24 & & & \\ \quad\quad \underline{+16- & 32} & & & \\ \quad\quad\quad +8- & 6 & & & \\ \quad\quad\quad \underline{+8- & 16} & & & \\ \quad\quad\quad\quad +10 & & & & \end{array}$$

Since the coefficient of x in $x - r$ is unity, the coefficient of the first term of each remainder is the coefficient of the next term to be obtained in the quotient. Further, it is not necessary to write the terms of the dividend as part of the remainder, nor the first term of the partial products.

The work thus becomes:

$$\begin{array}{rrrrrr|l} 5- & 6+8- & 24- & 6 & \underline{1-2} \\ \underline{-10} & & & & \\ +4 & & & & \\ \underline{\quad -8} & & & & \\ \quad +16 & & & & \\ \quad\quad \underline{-32} & & & & \\ \quad\quad\quad +8 & & & & \\ \quad\quad\quad\quad \underline{-16} & & & & \\ \quad\quad\quad\quad\quad +10 & & & & \end{array}$$

We may omit the first term of the divisor and write the work in the following more compact form:

$$\begin{array}{rrrrrr|l} 5- & 6+8- & 24- & 6 & \underline{-2} \\ \underline{-10-8-32-16} & & & & \end{array}$$

If we replace -2 by $+2$, we may add the partial products to the numbers in the dividend. Then, we have:

$$\begin{array}{rrrrrr|l} 5- & 6+ & 8- & 24- & 6 & \underline{2} \\ \underline{+10+} & 8+ & 32+ & 16 & & \\ 5+ & 4+ & 16+ & 8+ & 10 & \end{array}$$

The quotient is $5x^3 + 4x^2 + 16x + 8$, and the remainder is 10.

90. Rule for synthetic division. *To divide $f(x)$ by $x - r$, arrange $f(x)$ in descending powers of x, supplying all missing powers by putting in zeros as coefficients.*

Detach the coefficients, write them in a horizontal line and in the order $a_0, a_1, a_2, \cdots, a_n$.

Bring down the first coefficient a_0 ; multiply a_0 by r, and add the product to a_1 ; multiply this sum by r, and add the product to a_2. Continue this process ; the last sum is the remainder, and the preceding sums are the coefficients of the powers of x in the quotient, arranged in descending order.

PROOF OF RULE. This rule may be established by mathematical induction.

By long division,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_sx^{n-s} + a_{s+1}x^{n-s-1} + \cdots + a_n \,|\, x - r$$

$$\underline{\quad |a_0x^{n-1} + (a_1 + a_0r)x^{n-2} + \cdots + (a_{s-1} + ra_{s-2} + \cdots + r^{s-1}a_0)x^{n-s}}$$

$$\underline{a_0x^n - a_0rx^{n-1}}$$

$$\underline{(a_1 + a_0r)x^{n-1} + a_2x^{n-2}}$$

$$\underline{(a_1 + a_0r)x^{n-1} - (a_1r + a_0r^2)x^{n-2}}$$

. . . . . .

. . . . . .

We note that the coefficient of x^{n-2} in the quotient is formed according to the rule. Assume that the coefficients in the quotient down to that of x^{n-s} are formed according to the rule. On this hypothesis, proceed by long division to find the coefficient of x^{n-s-1} in the quotient. This may be exhibited as a continuation of the division above as follows :

$$(a_{s-1} + ra_{s-2} + \cdots + r^{s-1}a_0)x^{n-s+1} + a_sx^{n-s} + a_{s+1}x^{n-s-1} + \cdots + a_n$$

$$\underline{(a_{s-1} + ra_{s-2} + \cdots + r^{s-1}a_0)x^{n-s+1} - (ra_{s-1} + r^2a_{s-2} + \cdots + r^sa_0)x^{n-s}}$$

$$(a_s + ra_{s-1} + r^2a_{s-2} + \cdots + r^sa_0)x^{n-s} + a_{s+1}x^{n-s-1} + \cdots + a_n$$

This shows that if the coefficients in the quotient down to that of x^{n-s} are formed according to the rule, the coefficient of the next lower power is formed according to the rule. Hence, the rule is established.

EXERCISES

1. Divide $x^4 + 3x^3 - 5x + 3$ by $x - 4$ by synthetic division.

SOLUTION :

$$\begin{array}{r} 1 + 3 + 0 - 5 + 3 \,|\, 4 \\ \underline{\quad + 4 + 28 + 112 + 428} \\ 1 + 7 + 28 + 107 + 431 \end{array}$$

The quotient is $x^3 + 7x^2 + 28x + 107$ and the remainder is 431.

2. Divide $3x^3 + 5x^2 + 2x + 1$ by $x + 5$. (In this case $r = -5$.)

3. Divide $5x^3 - 6x + 3$ by $x + 2$.

4. Divide $7x^4 - 3x^2 - 2$ by $x + \frac{1}{2}$.

5. Divide $x^3 - 125$ by $x - 5$.

91. Graphs of polynomials. When the coefficients of $f(x)$ are real numbers, the march of the function for different values of x can be clearly presented by the use of the graphic methods explained in Arts. 27–28. To any value assigned to x, there corresponds one and only one value of the polynomial $f(x)$. This is sometimes expressed by saying that $f(x)$ is single valued. The fact that the graph of $f(x)$ is a continuous curve (see Art. 27) makes it of much service in the theory of equations.

EXERCISES

Construct the graphs of the following functions and locate their real zeros approximately (to within 0.5).

1. $f(x) = x^3 - 6x^2 + 11x - 6$.

As pointed out in Art. 89, synthetic division furnishes a convenient method of evaluating $f(x)$ for different values of x. Thus $f(0.5)$ is obtained as follows:

$$\begin{array}{r|l} 1 - 6 + 11 - 6 & 0.5 \\ \quad 0.5 - 2.75 + 4.125 & \\ \hline 1 - 5.5 + 8.25 - 1.875 & \end{array}$$

Hence, $f(0.5) = -1.875$.

In this way the following values are obtained:

$f(-2) = -60.$ $\quad f(1.5) = 0.375.$
$f(-1) = -24.$ $\quad f(2) = 0.$
$f(-0.5) = -13.125.$ $\quad f(2.5) = -0.375.$
$f(0) = -6.$ $\quad f(3) = 0.$
$f(0.5) = -1.875.$ $\quad f(4) = 6.$
$f(1) = 0.$ $\quad f(5) = 24.$

The graph is shown in Fig. 26; it presents to the eye the following facts:

(1) $f(x)$ has zeros at 1, 2, and 3.

(2) $f(x)$ is positive when $2 > x > 1$, and when $x > 3$.

(3) $f(x)$ is negative when $x < 1$ and when $3 > x > 2$.

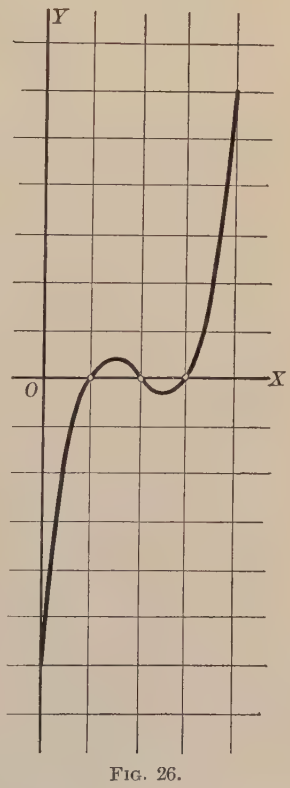

FIG. 26.

2. $x^3 - 6x^2 + 8x.$

3. $x^3 - 7x^2 + 14x - 8.$

4. $x^3 - 5x + 4.$

5. $x^4 - 2x^3 - 7x^2 + 8x + 12.$

6. $x^4 - 3x^2 + 5x - 6.$

7. $3x^4 - 4x^3 - 12x^2 + 3.$

8. $x^3 - 2x - 4.$

92. General equation of degree n. By equating to zero the general polynomial of the nth degree, we obtain what is known as the general equation of the nth degree in one unknown. That is to say,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

is the general equation of the nth degree.

The principal object of this chapter is to present methods which aid in determining exactly or approximately the real roots of special numerical * equations included under this type. It is largely for this purpose that we discuss the graphs of polynomials. The zeros of the polynomial are the *roots* of the equation formed by equating the polynomial to zero. The real roots of the equation may then be looked upon geometrically as the abscissas of the points of the X-axis where the graph of the polynomial cuts this axis.

93. Factor theorem. *If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of $f(x)$.*

Since a zero of $f(x)$ is a root of the equation $f(x) = 0$, this theorem follows directly from the corollary to the remainder theorem (Art. 88).

Exercise. State and prove the converse of the factor theorem.

94. Number of roots of an equation. *Every equation, $f(x) = 0$, of the nth degree has n roots and no more.*

To prove this proposition we assume the fundamental theorem that every equation, $f(x) = 0$, has at least one root. More explicitly, we assume that

There always exists at least one number, real or complex, which will satisfy an algebraic equation of the nth degree, whose coefficients are any real or complex numbers. †

* The term "numerical equations" is here used to indicate that the coefficients are not literal.

† This fundamental theorem was first proved by Gauss in 1797. For proof, see Fine's *College Algebra*, p. 588.

Let r_1 be a root, then (Art. 93), $x - r_1$ is a factor of $f(x)$ and

$$f(x) = (x - r_1) f_1(x), \tag{1}$$

where $f_1(x)$ is a polynomial of degree $n - 1$, beginning with the term $a_0 x^{n-1}$. By the theorem assumed, $f_1(x) = 0$ has at least one root. Let r_2 be a root; then

$$f_1(x) = (x - r_2) f_2(x),$$

and

$$f(x) = (x - r_1)(x - r_2) f_2(x), \tag{2}$$

in which $f_2(x)$ is a polynomial of degree $n - 2$, beginning with the term $a_0 x^{n-2}$. Continuing this process, we separate out n linear factors with a quotient a_0, so that

$$f(x) = a_0 (x - r_1)(x - r_2) \cdots (x - r_n) \tag{3}$$

where $r_1, r_2, \cdots, r_n$ are n roots of $f(x) = 0$.

If $f(x) = 0$ has another root different from any of these, let r denote such a root. Then, from (3),

$$a_0 (r - r_1)(r - r_2) \cdots (r - r_n) = 0.$$

But here we should have the product of factors equal to zero when no one of the factors is zero. As this is impossible (III, Art. 5), there are not more than n roots of $f(x) = 0$. Hence, every equation of the nth degree has n roots and no more Furthermore, every polynomial of the nth degree is the product of n linear factors. It is not, however, possible, in general, to determine these factors if n exceeds 4 (see Art. 106). Two or more of the n roots of $f(x) = 0$ may be equal to each other, in which case the equation is said to have multiple roots. If $(x - r)^m$ is a factor of $f(x)$, then $f(x) = 0$ is said to have m roots equal to r. Thus, $(x - 2)^5 = 0$ has five equal roots, and $(x - 1)^2 (x - 3)(x - 4)^3 = 0$ has two roots equal to 1, one equal to 3, and three equal to 4.

COROLLARY I. *If two polynomials of degrees not greater than n are equal to each other for more than n distinct values of the variable, the coefficients of like powers of the variable are equal.*

Let $$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = b_0 x^n + b_1 x^{n-1} + \cdots + b_n \tag{4}$$

for more than n values of x.

From (4),

$$(a_0 - b_0) x^n + (a_1 - b_1) x^{n-1} + \cdots + (a_n - b_n) = 0. \tag{5}$$

Then
$$a_0 - b_0 = 0,$$
$$a_1 - b_1 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n - b_n = 0.$$

For, if any coefficient in (5) were not equal to zero, we should have an equation of degree equal to or less than n with more than n roots, which is contrary to the theorem just proved.

Hence, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

COROLLARY II. *If two polynomials of degrees not greater than n are equal for more than n distinct values of the variable, they are equal for all values, and the equality is an identity.*

95. Graphs of $a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - r_1)(x - r_2)\cdots(x - r_n)$.

We assume that a_0, a_1, $\cdots$, a_n are real numbers, and further for convenience of expression that a_0 is positive, although this is not a necessary limitation. In Art. 91 the graphs of a few polynomials are plotted. Some important properties of these graphs appear when the polynomial is separated into linear factors. We cannot at this point make the treatment so complete as later, but we can well consider two important cases:

1. *When the factors $x - r_1$, $x - r_2$, $\cdots$, $x - r_n$ are all real and distinct.*

Arrange the factors so that $r_1 > r_2 > \cdots > r_{n-1} > r_n$. When $x > r_1$ all the factors are positive and the graph is above the X-axis.

FIG. 27.

When $r_1 > x > r_2$, one factor is negative and the graph is below the X-axis. When $r_2 > x > r_3$, two factors are negative, and the graph is again above the X-axis. Continuing this process, we see that the graph crosses the X-axis at the n points, $x = r_1$, $x = r_2$, $\cdots$,

$x = r_n$, and we obtain a general notion of the nature of the curve. See Fig. 27.

2. *When the factors are real but some of them repeated.*

To discuss the graph in this case, take for example,

$$f(x) = a_0(x - r_1)^2(x - r_2)(x - r_3)^5,$$

and let $$r_1 > r_2 > r_3.$$

Since the factors $x - r_2$ and $x - r_3$ occur to powers with odd exponents, it follows as above that the curve crosses the X-axis at $x = r_2$ and $x = r_3$. But it does not cross at $x = r_1$, since the sign of $f(x)$ is the same when $x > r_1$ as when $r_1 > x > r_2$, and the curve touches the axis at $x = r_1$. In general, if a factor $x - r$ occurs to a power with an odd exponent, the curve crosses the X-axis at $x = r$, while if it occurs to a power with an even exponent, it merely touches the X-axis at $x = r$. See Fig. 28.

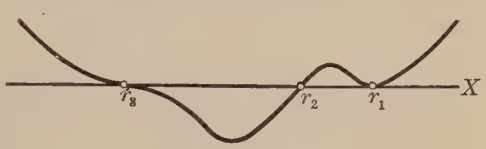

FIG. 28.

Another case is discussed in Art. 97, where imaginary factors occur.

96. Complex roots. *If a complex number $a + bi$ is a root of an equation $f(x) = 0$ with real coefficients, the conjugate complex number $a - bi$ is also a root.*

By the hypothesis, $a + bi$ satisfies the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0. \tag{1}$$

Put $a + bi$ for x in this equation and expand the powers of $a + bi$ by the binomial theorem. Represent the real part of this expansion by P and the imaginary part by iQ. Then

$$P + iQ = 0. \tag{2}$$

Whence, $$P = 0 \text{ and } Q = 0 \text{ (Art. 80).} \tag{3}$$

In the binomial expansion of $a + bi$ with any exponent, imaginary terms occur when and only when a term involves an odd power of bi as a factor. The result of substituting $a - bi$ instead of $a + bi$ is clearly obtained by replacing i by $-i$ in (2). We obtain thus $$P - iQ.$$

But, by (3), $P = 0$ and $Q = 0$;

hence, $P - iQ = 0$,

and $a - bi$ is a root of (1).

97. Graphs of $f(x)$ when some linear factors are imaginary. Since imaginary factors of $f(x)$ occur in conjugate pairs when the coefficients in $f(x)$ are real, it follows that in this case $f(x)$ may be regarded as the product of a_0, of real linear factors of the type $x - r$, and quadratic factors of the type

$$(x - a)^2 + b^2 = (x - a - bi)(x - a + bi),$$

where a, b, and r are real numbers. When all the roots of $f(x) = 0$ are real, the polynomial $f(x)$ is the product of real linear factors, but if $f(x) = 0$ has imaginary or complex roots, $f(x)$ contains quadratic factors of the type $(x - a)^2 + b^2$ which cannot be separated into real linear factors.

In Art. 95 the graph of $f(x)$ is discussed when the polynomial is the product of real linear factors, and it is shown that, corresponding to each linear factor $x - r$, the graph meets the X-axis at $x = r$. It should now be noted that

$$(x - a)^2 + b^2 > 0,$$

for all real values of x, and there is, therefore, corresponding to quadratic factors of $f(x)$, no intersection of the graph with the X-axis.

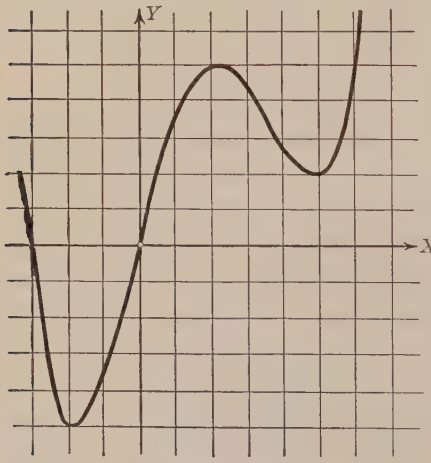

FIG. 29.

EXAMPLE: Graph

$$f(x) = x^4 - 7x^3 - 4x^2 + 78x$$
$$= x(x + 3)(x^2 - 10x + 26)$$
$$= x(x + 3)[(x - 5)^2 + 1].$$

Corresponding to the linear factors x and $x + 3$, the graph intersects the X-axis at $x = 0$ and $x = -3$ respectively (Fig. 29). Corresponding to the quadratic factor $x^2 - 10x + 26$ there is no intersection with the X-axis. (In Fig. 29 one horizontal space represents one unit, while one vertical space represents twenty units.)

EXERCISES

1. If $r_1, r_2, \cdots, r_n$ are roots of an equation, show that

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0$$

is the equation or an equivalent equation.

2. Form equations which have the following roots and no others.

(*a*) 2, 3, 5.

(*b*) $1 + 2i$, $1 - 2i$, when $i^2 = -1$.

(*c*) $1 + \sqrt{2}$, $1 - \sqrt{2}$, 3.

(*d*) $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{3}$, $-\sqrt{3}$.

(*e*) 1, -2, 3, 0.

(*f*) $2 + \sqrt{3}$, $2 - \sqrt{3}$, $-2 + \sqrt{3}$, $-2 - \sqrt{3}$.

3. By means of the theorem concerning the number of roots of $f(x) = 0$, show: (1) if $f(x) = 0$ be multiplied by a polynomial in x, the resulting equation has more roots than $f(x) = 0$; (2) if $f(x) = 0$ be divided by a polynomial in x, which is a factor of $f(x)$, the resulting equation will have fewer roots than $f(x) = 0$.

4. Plot the graphs of the following:

(*a*) $f(x) = (x - 1)^2(x - 3)^2$.

(*b*) $f(x) = x(x - 1)(x - 4)$.

(*c*) $f(x) = (x - 1)(x + 2)(x + 7)$.

(*d*) $f(x) = (x + 5)(x - 6)^2$.

(*e*) $f(x) = (x - 2)^2(x + 2)$.

5. Show that 3 and $\frac{1}{3}$ are double roots of

$$9x^5 - 51x^4 + 58x^3 + 58x^2 - 51x + 9 = 0,$$

and find the other root.

Separate the following polynomials into real linear and quadratic factors and plot the graphs.

6. $x^3 - 1$. **7.** $x^3 + 1$. **8.** $x^4 - 1$.

9. $x^6 - 1$. **10.** $x^4 + 4x^3 + 3x^2 - 4x - 4$.

11. Show that an equation $f(x) = 0$ of odd degree and with real coefficients has an odd number of real roots.

98. Transformations of equations. Frequently an equation can be solved more readily, or its properties can be discussed better, by transforming it into an equation whose roots are related to those of the given equation in some specified manner. For example, in solving numerical equations for their real roots (Arts.

102–105), we shall have use for the following transformations of $f(x)=0$:

$$x=\frac{x'}{m},\ x=-x',\text{ and } x=x'+h.$$

If we make $x=\frac{x'}{m}$ (or $x'=mx$) in $f(x)=0$, we obtain an equation in x' whose roots are m times those of $f(x)=0$. If we make $x=-x'$ in $f(x)=0$, we obtain an equation in x' whose roots are equal in absolute value but opposite in sign to those of $f(x)=0$. If we make $x=x'+h$ in $f(x)=0$, we obtain an equation in x' each of whose roots is less by h than the corresponding root of $f(x)=0$.

These transformations can be performed rapidly by means of the following rules:

1. *To obtain an equation each of whose roots is m times a corresponding root of $f(x)=0$: multiply the successive coefficients beginning with that of x^{n-1} by m, m^2, m^3, $\cdots$, respectively.* *

For example, to find the equation each of whose roots is double the roots of the equation $x^4-4x^3+3x^2+1=0$, we make $m=2$, and obtain

$$x^4-2(4x^3)+2^2(3x^2)+2^3(0\cdot x)+2^4=0,$$
$$x^4-8x^3+12x^2+16=0.$$

To establish this rule, substitute $x=\frac{x'}{m}$ in

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0. \quad (1)$$

The result of this substitution is

$$a_0\left(\frac{x'}{m}\right)^n+a_1\left(\frac{x'}{m}\right)^{n-1}+a_2\left(\frac{x'}{m}\right)^{n-2}+\cdots+a_{n-1}\left(\frac{x'}{m}\right)+a_n=0, \quad (2)$$

or $$a_0x'^n+ma_1x'^{n-1}+m^2a_2x'^{n-2}+\cdots+m^{n-1}a_{n-1}x'+m^na_n=0, \quad (3)$$

after multiplying by the constant m^n. The rule is thus established.

2. *To obtain an equation each of whose roots is equal in absolute value to a root of $f(x)=0$, but opposite in sign: change the signs of the odd degree terms in $f(x)=0$.*

* In carrying out this rule any missing power of x should be supplied with zero as a coefficient.

For example, the roots of the equation

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$$

are 2, 4, -1, -3, and the equation with roots -2, -4, 1, 3 is

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0.$$

The rule follows at once from rule 1, by making $m = -1$.

3. *To obtain an equation each of whose roots is less by h than a corresponding root of a given equation $f(x) = 0$: divide $f(x)$ by $x - h$ and indicate the remainder by R_n. Divide the quotient by $x - h$, and indicate the remainder by R_{n-1}. Continue this process to n divisions. The last quotient, a_0, and the remainders, R_1, R_2, $\cdots$, R_n are the coefficients of the transformed equation. The new equation is then,*

$$a_0x'^n + R_1x'^{n-1} + R_2x'^{n-2} + \cdots R_{n-1}x' + R_n = 0.$$

The division should be performed by the method of synthetic division.

For example, find the equation each of whose roots is less by 2 than the roots of the equation

$$x^3 - 4x^2 - 3x + 2 = 0.$$

The work is as follows:

$$\begin{array}{rrrr|l} 1 & -4 & -3 & +2 & 2 \\ & +2 & -4 & -14 & \\ \hline 1 & -2 & -7 & -12 & R_3 = -12, \\ & +2 & 0 & & \\ \hline 1 & +0 & -7 & & R_2 = -7, \\ & & 1 & & \\ \hline 1 & +1 & & & R_1 = 1, \\ & & & & a_0 = 1. \end{array}$$

The required equation is

$$x^3 + x^2 - 7x - 12 = 0.$$

To establish the rule, substitute $x = x' + h$ in

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \qquad (1)$$

This gives the equation in x'

$$a_0(x' + h)^n + a_1(x' + h)^{n-1} + \cdots + a_{n-1}(x' + h) + a_n = 0 \qquad (2)$$

whose roots are less by h than those of (1). Expanding the binomial powers and arranging in powers of x', we may present the result in the form

$$a_0x'^n + A_1x'^{n-1} + A_2x'^{n-2} + \cdots + A_{n-1}x' + A_n = 0. \qquad (3)$$

If in (3), we make $x' = x - h$, we obtain

$$a_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} + \cdots + A_{n-1}(x-h) + A_n = 0 \qquad (4)$$

which is the same as equation (1) arranged in powers of $x - h$. From the form of equation (4), it follows that A_n is the remainder when $f(x)$ is divided by $x - h$; A_{n-1} is the remainder when the quotient of the last-named division is divided by $x - h$; continuing this process to n divisions, A_1 is the last remainder, and a_0 is the last quotient. That is,

$$\begin{aligned} A_n &= R_n, \\ A_{n-1} &= R_{n-1}, \\ &\cdots\cdots \\ A_1 &= R_1, \end{aligned}$$

which establishes the rule.

EXERCISES

Obtain equations whose roots are equal to the roots of the following equations multiplied by the number opposite.

1. $x^3 + 2x^2 - 7x - 1 = 0.$ (5)
2. $x^3 - \frac{x^2}{5} - \frac{8}{25} = 0.$ (5)
3. $x^4 - 10x^2 - 3x - 2 = 0.$ (−1)
4. $x^4 - 3x^2 + 10 = 0.$ (−2)

Obtain equations whose roots are equal to the roots of the following equations multiplied by the smallest number which will make all the coefficients integers and that of the highest power unity.

5. $x^3 - 2x^2 + \frac{1}{3}x - 10 = 0.$
6. $2x^4 + 3x^3 - 6x^2 + 5x - 1 = 0.$
7. $3x^3 + 4 = 0.$
8. $x^3 + x^2 + \frac{x}{6} - \frac{83}{36} = 0.$

9. Obtain equations whose roots are equal in absolute value but opposite in sign to the roots of equations given in Exs. 1–4.

Obtain equations whose roots are equal to the roots of the following equations diminished by the number opposite.

10. $2x^4 - 3x^2 + 4x - 5 = 0$. (2)

SOLUTION: We apply transformation 3, Art. 98. By synthetic division, this gives

$$\begin{array}{rrrrrl|l}
2 + & 0 - & 3 + & 4 - & 5 & & 2 \\
 & + & 4 + & 8 + & 10 + & 28 & \\
\hline
2 + & 4 + & 5 + & 14 + & 23 & & R_4 = 23, \\
 & + & 4 + & 16 + & 42 & & \\
\hline
2 + & 8 + & 21 + & 56 & & & R_3 = 56, \\
 & + & 4 + & 24 & & & \\
\hline
2 + & 12 + & 45 & & & & R_2 = 45, \\
 & + & 4 & & & & \\
\hline
2 + & 16 & & & & & R_1 = 16, \\
 & & & & & & a_0 = 2.
\end{array}$$

Hence, $2x^4 + 16x^3 + 45x^2 + 56x + 23 = 0$ is the required equation.

11. $x^2 - 7x + 6 = 0$. (3)

12. $x^3 - 7x + 7 = 0$. (1)

13. $x^3 - 27x - 36 = 0$. (3)

14. $x^5 - 6x^4 + 7.4x^3 + 7.92x^2 - 17.872x - 0.79232 = 0$. (1.2)

15. $2x^4 + 16x^3 + 45x^2 + 56x + 23 = 0$. (−2)

99. Descartes's rule of signs. In a polynomial arranged in descending powers of x, if two successive terms differ in sign there is said to be a **variation** in sign. For example,

$$x^4 - 4x^3 + 3x^2 + 4x - 5$$

has 3 variations of sign as is shown more clearly by writing down the signs $+ - + + -$. Multiply this polynomial by $x - 1$.

There results $x^5 - 5x^4 + 7x^3 + x^2 - 9x + 5$

with 4 variations of sign. This last polynomial has one more positive zero (see Art. 95) than the first. If increasing the number of positive zeros of a polynomial always increases the number of variations in sign by at least one, then the number of positive roots is never greater than the number of variations of sign.

THEOREM. *The number of positive roots of an equation $f(x) = 0$ does not exceed the number of variations of sign of $f(x)$, nor does the number of negative roots exceed the number of variations of sign of $f(-x)$.*

This is Descartes's rule of signs.

The part which relates to positive roots will be established by showing that whenever a positive root r is introduced into an equation, there is added at least one variation of sign. Let $f(x) = 0$ be an equation of degree m. It is only necessary to show that $(x - r)f(x)$ has at least one more variation of sign than $f(x)$. Group the terms of $f(x)$ between consecutive variations of sign in brackets. In general, for a function of degree m, we have

$$\begin{aligned} f(x) = {} & [b_0x^m + b_1x^{m-1} + \cdots + b_px^{m-p}] \\ & - [b_{p+1}x^{m-p-1} + b_{p+2}x^{m-p-2} + \cdots + b_qx^{m-q}] \\ & + [b_{q+1}x^{m-q-1} + \cdots] \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \pm [b_{m-v}x^v + b_{m-v+1}x^{v-1} + \cdots + b_m], \end{aligned}$$

where b_0, b_1, b_2, $\cdots$ b_m are positive.

Multiplying this function first by x, then by $-r$, and adding, we obtain

$$\begin{aligned} (x - r)f(x) = {} & [b_0x^{m+1} + (b_1 - b_0r)x^m \pm \cdots] \\ & - [(b_{p+1} + rb_p)x^{m-p} \pm \cdots] \\ & + [(b_{q+1} + rb_q)x^{m-q} \pm \cdots] \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \pm [(b_{m-v} + rb_{m-v-1})x^{v+1} \pm \cdots] \\ & \mp rb_m. \end{aligned}$$

It will be noticed that the coefficients of the first terms in the several brackets, that is, b_0, $(b_{p+1} + rb_p)$, $(b_{q+1} + rb_q)$, $\cdots$, are all positive. Hence, the signs between the brackets remain unchanged. The signs within the brackets are uncertain, but however they may occur there is at least one variation between the first term of one bracket and the first term of the next bracket. Hence, as far as the terms in the brackets are concerned the number of variations remains the same or is increased. But there is added the variation caused by the term $\mp rb_m$ whose sign differs from the sign of the last bracket. Therefore, there is at least one more variation in $(x - r)f(x)$ than in $f(x)$.

The part of Descartes's rule which relates to negative roots follows from the fact that the roots of $f(-x) = 0$ are equal in absolute value but opposite in sign to the roots of $f(x) = 0$.

EXERCISES

Find the maximum number of positive and of negative roots, and any other information about the nature of the roots of

1. $x^3 + 5x - 7 = 0$.

SOLUTION: There is one variation in sign, hence, there is not more than one positive root. $f(-x) = -x^3 - 5x - 7$ with no variation in sign, hence, there are no negative roots. Since there are three roots of the equation, two are imaginary and one positive.

2. $3x^4 + x^2 + 2 = 0$.

3. $x^5 + 1 = 0$.

4. $x^3 + 3x^2 + 1 = 0$.

5. $x^6 + 4x^2 + x = 0$.

6. $x^n - 1 = 0$ (n odd).

7. $x^n - 1 = 0$ (n even).

8. Given that the roots of $5x^3 - 3x^2 - 4x + 11 = 0$ are all real, determine the signs of the roots.

9. Show that the equation $7x^6 - 2x^2 - 2x + 4 = 0$ has at least two imaginary roots.

10. Show that the equation $x^8 + 5x^3 + 4x - 10 = 0$ has six and only six imaginary roots.

100. Location of roots by graph. If the real roots of an equation $f(x) = 0$ are greater than a and less than b, these roots are said to be contained in the interval a to b along the X-axis. The number a is a lower limit and the number b is an upper limit of the interval.

We are concerned with the graph of $f(x)$ chiefly throughout an interval along the X-axis which contains the roots of $f(x) = 0$. To avoid the labor of plotting the graph outside of this interval, it is desirable, in evaluating $f(x)$ for $x = b$ by synthetic division, to know whether b is greater than any root. The following criterion will be found helpful.

If all the sums are positive in the synthetic division by $x - b$ (b positive), then b is greater than any root. For, a greater number than b would make the sums still greater. For example, to show that 6 is greater than any real root of

$$x^4 - 5x^3 + 3x^3 - 42x - 50 = 0,$$

we divide by $x - 6$ by synthetic division,

$$\begin{array}{r|l} 1 - 5 + 3 - 42 - 50 & 6 \\ \quad 6 + 6 + 54 + 72 & \\ \hline 1 + 1 + 9 + 12 + 22 & \end{array}$$

and we observe that a number greater than 6 would increase each sum.

To find a lower limit of the negative roots of $f(x) = 0$, it is only necessary to find as above, by synthetic division, an upper limit of the positive roots of $f(-x) = 0$.

After the graph of $f(x)$ is plotted throughout an interval which contains the roots, at least the approximate values of the real roots of $f(x) = 0$ are presented geometrically. The following principle aids in locating roots during the process of plotting the graph.

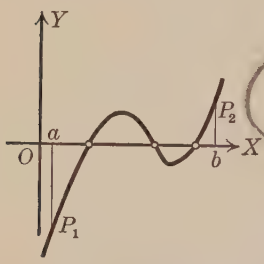

FIG. 30.

If $f(a)$ and $f(b)$ have contrary signs, the equation $f(x) = 0$ has at least one real root between a and b.

For the points P_1 and P_2 (Fig. 30) which correspond to $x = a$ and $x = b$ are on opposite sides of the X-axis, and any continuous curve connecting P_1 and P_2 crosses the X-axis at least once between a and b. Since, to every intersection of the graph with the X-axis there corresponds a real root of the equation (Art. 95), we assume this principle.

EXERCISES

1. Show that $x^3 + 8x - 7 = 0$ has a real root between 0 and 1, and that the other roots are imaginary.

2. Show that -1 is a lower limit of the roots of

$$x^4 - 5x^3 + 3x^2 - 42x - 50 = 0.$$

3. Show that 2 is an upper limit of the roots of $x^4 - 2x^3 + 3x^2 - 5x + 1 = 0$.

Find the integral part of each real root of

4. $x^3 + x^2 - 2x - 1 = 0$.

5. $x^3 + 2x + 5 = 0$.

6. $x^3 - 2x - 5 = 0$.

7. $x^3 + 2x - 5 = 0$.

8. $x^4 - 12x^3 + 12x - 3 = 0$.

9. $8x^3 - 36x^2 + 46x - 15 = 0$.

10. $x^3 - 3x^2 - 4x + 11 = 0$.

11. $x^3 - 2x - 5 = 0$.

101. Equation in the p-form. An equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0, \tag{1}$$

where $p_1, p_2, \cdots, p_n$ are any numbers, may be called the p-form of the equation of the nth degree.

The general equation,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \tag{2}$$

can clearly be reduced to the p-form by dividing its members by a_0. The p-form is more convenient for the statement of certain theorems (Art. 102) than the form (2).

Exercise. Reduce $6x^3 - 3x^2 + 2x - 5 = 0$ to the p-form.

102. Rational roots. *Any rational root of an equation $f(x) = 0$ in the p-form with integral coefficients is an integer and an exact divisor of p_n.*

To prove this theorem, suppose, if possible, that $\frac{a}{b}$ is a root of $f(x) = 0$, where $\frac{a}{b}$ is a fraction in its lowest terms. Then,

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0. \tag{1}$$

Multiplying (1) by b^{n-1}, we obtain

$$\frac{a^n}{b} + p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1} = 0,$$

or

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}). \tag{2}$$

All the terms of the right-hand member of (2) are integers, while the left-hand member is a fraction in its lowest terms. Hence, the hypothesis that $\frac{a}{b}$ is a root leads to an absurdity.

Next, suppose that c is a root, where c is an integer. Then,

$$c^n + p_1c^{n-1} + p_2c^{n-2} + \cdots + p_{n-1}c + p_n = 0. \tag{3}$$

Transposing p_n and dividing through by c, we obtain

$$c^{n-1} + p_1c^{n-2} + p_2c^{n-3} + \cdots + p_{n-1} = -\frac{p_n}{c}. \tag{4}$$

Each term of the left-hand member of (4) is an integer. Hence, $\frac{p_n}{c}$ is an integer; that is, c is an exact divisor of p_n.

*To obtain the rational roots of an equation in p-form with integral coefficients, it is only necessary to test whether the integers which are the exact divisors of p_n satisfy the equation.**

EXERCISES

Find the rational roots of the following equations.

1. $x^3 - 9x^2 + 23x - 15 = 0$.

SOLUTION: By Descartes's rule of signs, this equation has no negative roots. Hence, we need try only 1, 3, 5, and 15. By synthetic division,

$$\begin{array}{r|l} 1 - 9 + 23 - 15 & 1 \\ \quad +1 - 8 + 15 & \\ \hline 1 - 8 + 15 + 0 & \end{array}$$

The depressed equation is $x^2 - 8x + 15 = (x - 5)(x - 3) = 0$. Hence, 1, 3, and 5 are the roots.

2. $108x^3 - 54x^2 + 45x - 13 = 0$.

SOLUTION: In the p-form this equation is

$$x^3 - \tfrac{1}{2}x^2 + \tfrac{5}{12}x - \tfrac{13}{108} = 0. \qquad (1)$$

Transform (1) into an equation whose roots are 6 times those of (1). This gives

$$x^3 - 3x^2 + 15x - 26 = 0. \qquad (2)$$

The rational roots of (2) divided by 6 give the rational roots of (1). By Descartes's rule, (2) has no negative roots. Hence, we need try only 1, 2, 13, 26. Depressing the equation,

$$\begin{array}{r|l} 1 - 3 + 15 - 26 & 1 \\ \quad +1 - 2 + 13 & \\ \hline 1 - 2 + 13 - 13 & \end{array}$$

Hence, 1 is not a root.

$$\begin{array}{r|l} 1 - 3 + 15 - 26 & 2 \\ \quad +2 - 2 + 26 & \\ \hline 1 - 1 + 13 + 0 & \end{array}$$

The depressed equation $x^2 - x + 13 = 0$ has no rational roots. Hence, 2 is the only rational root of (2) and $\frac{1}{3}$ is the only rational root of (1).

3. $x^4 - 15x^2 + 10x + 24 = 0$.

4. $x^3 - 4x^2 + 2x - \frac{1}{8} = 0$.

5. $x^3 + 3x^2 - 4x - 12 = 0$.

6. $2x^3 + x^2 + 2x + 1 = 0$.

---

* If p_n is a large number, this method is too long to be practical.

7. $3x^3 + 8x^2 + x - 2 = 0$.

8. $4x^3 - 8x^2 + 5x - 1 = 0$.

9. $x^4 - x^3 - 8x^2 - 14x + 60 = 0$.

10. $10x^4 + 17x^3 - 16x^2 + 2x = 0$.

11. $4x^3 - 12x^2 + 27x - 19 = 0$.

12. $x^3 - 4x^2 - 17x + 60 = 0$.

13. $x^4 - 45x^2 + 40x + 84 = 0$.

14. $8x^3 - 2x^2 - 4x + 1 = 0$.

15. $24x^4 - 4x^3 - 2x^2 - \frac{13x}{24} + \frac{1}{4} = 0$.

16. $108x^3 - 54x^2 + 45x - 13 = 0$.

17. $x^5 - 8x^4 + 15x^3 + 20x^2 - 76x + 48 = 0$.

103. Irrational roots. Horner's method. The irrational roots of a numerical equation can be obtained to any desired number of decimal places by a method of approximation called Horner's method. The method can be best explained by first applying it to an example. In case an equation has some rational roots, it should always be depressed by removing such roots before considering irrational roots.

EXAMPLE: Find the real roots of

$$x^4 - 2x^3 + 4x^2 - 15x + 14 = 0. \qquad (1)$$

1. Test for rational roots as in Art. 102. It results that 2 is the only rational root.

$$\begin{array}{r} 1 - 2 + 4 - 15 + 14 \,\big|\, \underline{2} \\ \quad + 2 + 0 + \quad 8 - 14 \\ \hline 1 + 0 + 4 - \quad 7 + \quad 0 \end{array}$$

The depressed equation is

$$x^3 + 4x - 7 = 0. \qquad (2)$$

2. Test for the interval which contains the real roots. From Descartes's rule, equation (2) has not more than one positive root, and it has no negative root. Furthermore, 2 is greater than any root (Art. 100).

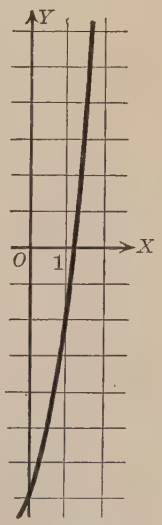

FIG. 31.

3. Plot $x^3 + 4x - 7$ from $x = 0$ to $x = 2$.

The graph (Fig. 31) shows that 1 is the first figure of the root.

4. Transform to diminish roots by 1; or graphically, change the origin to the point marked 1. The numerical work is as follows:

$$\begin{array}{r|l} 1+0+4-7 & 1 \\ \quad +1+1+5 & \\ \hline 1+1+5 & -2 \\ \quad +1+2 & \\ \hline 1+2 & +7 \\ \quad +1 & \\ \hline 1+3 & \end{array}$$

The first transformed equation is then

$$x_1^3 + 3x_1^2 + 7x_1 - 2 = 0. \tag{3}$$

This equation has a root between 0 and 1, since (2) has a root between 1 and 2. By evaluating $f(x_1) = x_1^3 + 3x_1^2 + 7x_1 - 2$ for successive tenths (0.0, 0.1, 0.2, ···, 0.9), we find that this function is negative when $x_1 = 0.2$ and positive when $x_1 = 0.3$. Hence, (3) has a root between 0.2 and 0.3. An approximation to this root is given by neglecting the second and third degree terms in (3) and solving

$$7x_1 - 2 = 0.$$

The root of this equation between 0 and 1 is $x_1 = 0.2\cdots$. It is important to observe from the graph of $f(x)$ that the sign of the known term in each transformed equation is to be the same as that of the original equation after the rational roots have been removed.

Transforming (3) into an equation whose roots are less by 0.2, we have

$$\begin{array}{rrrr|l} 1 & 3 & 7 & -2 & 0.2 \\ & 0.2 & 0.64 & 1.528 & \\ \hline 1 & 3.2 & 7.64 & -0.472 & \\ & 0.2 & 0.68 & & \\ \hline 1 & 3.4 & 8.32 & & \\ & 0.2 & & & \\ \hline 1 & 3.6 & & & \end{array}$$

or

$$x_2^3 + 3.6x_2^2 + 8.32x_2 - 0.472 = 0 \tag{4}$$

as the second transformed equation. The root of equation (4) which we seek lies between 0 and 0.1. Neglecting powers of x_2 higher than the first, it appears from the equation

$$8.32x_2 - 0.472 = 0$$

that x_2 lies between 0.05 and 0.06. That the root is in this interval may be tested by evaluating $x_2^3 + 3.6\,x_2^2 + 8.32\,x_2 - 0.472$ for $x_2 = 0.05$ and 0.06.

Transforming (4) by synthetic division into an equation whose roots are less by 0.05, we obtain

$$x_3^3 + 3.75\,x_3^2 + 8.6875\,x_3 - 0.046875 = 0. \qquad (5)$$

Neglecting powers of x_3 higher than the first, it appears from the equation

$$8.6875\,x_3 - 0.046875 = 0$$

that x_3 lies between 0.005 and 0.006.

Transforming (5) by synthetic division into an equation whose roots are less by 0.005, we have

$$x_4^3 + 3.765\,x_4^2 + 8.725075\,x_4 - 0.003343625 = 0. \qquad (6)$$

The root of this equation between 0 and 0.001 can be obtained at least as far as the first figure by neglecting powers of x_4 above the first. This gives

$$x_4 = 0.0003^+.$$

Transforming (6) into an equation whose roots are less by 0.0003, we obtain

$$x_5^3 + 3.7659\,x_5^2 + 8.72733427\,x_5 - 0.000725763623 = 0.$$

The root of this equation between 0 and 0.0001 can be obtained at least so far as the first significant figure by neglecting powers of x_5 above the first. This gives

$$x_5 = 0.00008^+.$$

Taking the sum of successive diminutions of the roots of (2), we obtain as the approximate value of the root sought

$$x = 1.25538^+.$$

The preceding work of transformation may be compactly arranged as follows :

1	**0**	**4**	**− 7**	**| 1.**
	1	1	5	
1	1	5	− 2	
	1	2		
1	2	7		
	1			
1	**3**	**7**	**− 2**	**| 0.2**
	0.2	0.64	1.528	
1	3.2	7.64	− 0.472	
	0.2	0.68		
1	3.4	8.32		
	0.2			
1	**3.6**	**8.32**	**− 0.472**	**| 0.05**
	0.05	0.1825	0.425125	
1	3.65	8.5025	− 0.046875	
	0.05	0.1850		
1	3.70	8.6875		
	0.05			
1	**3.75**	**8.6875**	**− 0.046875**	**| 0.005**
	0.005	0.018775	0.043531375	
1	3.755	8.706275	− 0.003343625	
	0.005	0.01880		
1	3.760	8.725075		
	0.005			
1	**3.765**	**8.725075**	**− 0.003343625**	**| 0.0003**
	0.0003	0.00112959	.002617861377	
1	3.7653	8.72620459	− 0.000725763623	
	0.0003	0.00112968		
1	3.7656	8.72733427		
	0.0003			
1	**3.7659**	**8.72733427**	**− 0.000725763623**	**| 0.00008**
	0.00008	0.0003012784	0.000698210843872	
1	3.76598	8.7276355484	− 0.000027552779128	

The heavy type indicates the successive transformed equations. The process can evidently be continued to find the root to any required number of decimal places.

If a root of an equation is known to be small, one important point to note is that such a root can, in general, be well estimated by dividing the known term, with its sign changed, by the coefficient of the first degree term. The coefficient of the first degree term is, for this reason, sometimes called the **trial divisor** in obtaining approximate roots. A still better estimate of a root can,

in general, be obtained by dropping terms of degree higher than the second, and solving the quadratic.

When an equation has more than one irrational root, each is treated separately as we have treated the single irrational root in this example.

If two roots of an equation $f(x)=0$ are nearly equal, their separation may become laborious, but the separation may be accomplished by assigning values to x sufficiently near each other in plotting the graph of $f(x)$. For example,

$$4x^3 - 24x^2 + 44x - 23 = 0,$$

has two roots between 2 and 3. By assigning successively the values $x = 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3$, in plotting the graph we find that one of these roots is between 2.2 and 2.3, while the other is between 2.8 and 2.9.

104. Negative roots. The negative roots of $f(x)=0$ are obtained by finding the positive roots of $f(-x)=0$, and changing their signs. It is therefore sufficient to discuss the method of obtaining positive roots.

105. Summary. In solving a numerical equation $f(x)=0$ for all its real roots, the following rules may be found helpful in systematizing the work:

1. *Test for rational roots; and if any exist, depress the equation by removing the corresponding factors.*

2. *Determine an interval which contains the irrational roots.*

3. *Plot the depressed polynomial throughout this interval to locate the roots.*

4. *Apply Horner's method to find the irrational roots. To do this, fix the attention upon some positive root whose location is known to be between two consecutive integers. Obtain by synthetic division (Art. 89) an equation whose roots are less than those of the given equation by the smaller of these two integers. The new equation has a root between 0 and 1. Locate this root between two successive tenths; and decrease the roots by the smaller of these tenths. The equation thus obtained has a root between 0 and 0.1. Locate this root between two successive hundredths, and again decrease the*

roots by the smaller of these hundredths. Continue this process to any required number of decimal places.

Add together all the diminutions of the roots to obtain the required root.

If more than one root is contained between two consecutive integers, separate them by means of the location principle.

5. *Treat negative roots in the same manner as positive roots after changing* $f(x) = 0$ *into* $f(-x) = 0$.

EXERCISES AND PROBLEMS

Find a positive root of each of the following equations to two decimal places.

1. $x^3 - 3x^2 - 2x + 5 = 0$.

2. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$.

3. $x^5 + 12x^4 + 59x^3 + 150x^2 + 201x - 207 = 0$.

4. $x^3 - 100 = 0$.

5. $x^5 - 1000 = 0$.

Find the rational roots, and the value of irrational real roots to two decimal places, of the following equations.

6. $x^3 - 3x^2 - 2x + 6 = 0$.

7. $x^3 + 30x - 420 = 0$.

8. $x^3 - 3x^2 - 4x + 13 = 0$.

9. $x^3 + 3x^2 + 4x + 5 = 0$.

10. $x^3 + 4x^2 - 5x - 20 = 0$.

11. $3x^4 - 2x^3 - 21x^2 - 4x + 11 = 0$.

12. $2x^4 - 12x^3 + 12x - 3 = 0$.

13. $x^3 - 3x - 1 = 0$.

14. $x^4 - 3x^3 + 3 = 0$.

15. $x^4 + 4x^2 + 4x + 3 = 0$.

16. An open box is made of a rectangular piece of tin 10 inches by 20 inches by cutting equal squares from the corners and turning up the sides. Find (to two decimal places) the side of a square cut out if the volume of the box is 187 cubic inches.

17. A piece of property can be bought for \$7550 cash or \$8000, payable in four equal annual instalments of \$2000 each, the first instalment being paid at once, and the remaining instalments at the ends of 1, 2, and 3 years. What yearly rate of interest compounded annually gives the two offers equal present values?

18. A sphere of yellow pine 1 foot in diameter floating in water sinks to a depth x given by $2x^3 - 3x^2 + 0.9657 = 0$.

Find the depth to 3 significant figures.

19. A sphere of ice 1 foot in diameter floating in water sinks to a depth x given by the equation $2x^3 - 3x^2 + 0.693 = 0$.

Find the depth to 3 significant figures.

20. A cork sphere 1 foot in diameter floating on water sinks to a depth x given by the equation

$$2x^3 - 3x^2 + 0.24 = 0.$$

If the sphere is 2 feet in diameter, the immersed depth is given by

$$2x^3 - 6x^2 + 1.92 = 0.$$

Find the depths to 2 significant figures.

21. The width of the strongest beam which can be cut from a log 12 inches in diameter is given by the positive irrational root of the equation

$$x^3 - 144x + 665 = 0.$$

Find the width to 3 significant figures.

22. The speed in feet per second of a 1-inch manila rope transmitting 4 horse power, under a tension of 300 pounds on the tight side, is given by the equation

$$v^3 - 19200v + 211200 = 0.$$

Find the velocity to 3 significant figures.

23. The diameter of a water pipe whose length is 200 feet, and which is to discharge 100 cubic feet per second under a head of 10 feet, is given by the real root of the equation

$$x^5 - 38x - 101 = 0.$$

Find the diameter to 3 significant figures.

(Merriman and Woodward, *Higher Mathematics*, p. 13.)

24. The algebraic treatment of the trisection of an angle whose sine is a involves the solution of the cubic equation

$$4x^3 = 3x - a.$$

The unknown, x, is the sine of one third the given angle. When $a = \frac{1}{2}\sqrt{2}$, find x to 3 significant figures.

25. A vat in the form of a rectangular parallelopiped is $8 \times 10 \times 12$ feet. If the volume is increased 500 cubic feet by equal elongations of the dimensions, find elongations in feet to two decimal places.

26. In Problem 25, if the volume is increased by elongations proportional to the dimensions, find each elongation.

27. From the American Report on Wholesale Prices, Wages, and Transportation, for 1891, the median wage is given in dollars by $\frac{1}{4}$ of a value of x in the equation

$$2561\tfrac{1}{2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

where $a_0 = 6972\frac{91}{128}$, $a_1 = -657\frac{5}{48}$, $a_2 = -33\frac{43}{64}$, $a_3 = \frac{197}{48}$, $a_4 = -\frac{5}{128}$. Find the median wage correct to mills.

106. Algebraic solution of equations. In Arts. 102–105, methods are discussed by which we obtain approximately the real roots of numerical equations. We turn now to a brief consideration of equations with literal coefficients.

Solving such an equation consists in obtaining an expression in terms of the coefficients which satisfies the equation. In other words, it consists in finding a formula which gives the roots in terms of the coefficients. For example, the roots of the typical quadratic

$$ax^2 + bx + c = 0,$$

are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots of an equation are functions of the coefficients, and it is important to inquire into the character of these functions. The solution is said to be an **algebraic solution**, if these functions of the coefficients involve no operations except a finite number of additions, subtractions, multiplications, divisions, and extractions of roots.

The algebraic solution of an equation is often called the **solution by radicals.**

In Arts. 107, 108, the general cubic

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

and the general biquadratic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

are solved by radicals.

The algebraic solution of the general fifth degree equation

$$a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$$

engaged the attention of mathematicians during the eighteenth and the first quarter of the nineteenth century. In 1826 Abel proved that the typical fifth degree equation has no algebraic solution. Since that time a branch of mathematics, known as the theory of substitution groups, has been much developed. While a treatment of substitution groups is beyond the scope of this book, it

may be stated that, by means of this theory, it is shown that no typical equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

has an algebraic solution if n exceeds 4; and necessary and sufficient conditions that an equation has an algebraic solution are established.

107. The cubic equation. The general cubic equation is

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0. \tag{1}$$

By making

$$x = y - \frac{a_1}{3\,a_0}, \tag{2}$$

equation (1) is transformed into

$$a_0y^3 + \left(a_2 - \frac{a_1^2}{3\,a_0}\right)y + \frac{2\,a_1^3}{27\,a_0^2} - \frac{a_1a_2}{3\,a_0} + a_3 = 0,$$

or

$$y^3 + \frac{3\,a_0a_2 - a_1^2}{3\,a_0^2}\,y + \frac{2\,a_1^3}{27\,a_0^3} - \frac{a_1a_2}{3\,a_0^2} + \frac{a_3}{a_0} = 0, \tag{3}$$

which has no term of the second degree.

Let

$$3\,H = \frac{3\,a_0a_2 - a_1^2}{3\,a_0^2}, \tag{4}$$

and

$$G = \frac{2\,a_1^3}{27\,a_0^3} - \frac{a_1a_2}{3\,a_0^2} + \frac{a_3}{a_0}. \tag{5}$$

Then (3) takes the form,

$$y^3 + 3\,Hy + G = 0. \tag{6}$$

Now assume

$$y = u^{\frac{1}{3}} + v^{\frac{1}{3}}, \tag{7}$$

and

$$-H^3 = uv. \tag{8}$$

From (6), (7), and (8),

$$-G = u + v. \tag{9}$$

Eliminating v from (8) and (9), we have

$$u^2 + Gu - H^3 = 0, \tag{10}$$

and solving this quadratic in u, we find for a solution,

$$u = \frac{-G + \sqrt{G^2 + 4\,H^3}}{2}. \tag{11}$$

From (8) and (11) we have

$$v = -\frac{H^3}{u} = \frac{-G - \sqrt{G^2 + 4\,H^3}}{2}. \tag{12}$$

The double sign before the radical in the solution of the quadratic in u is omitted because taking the negative sign before the radical would simply interchange the values of u and v. Since

$$y = u^{\frac{1}{3}} + v^{\frac{1}{3}},$$

the three values of y are :

$$\left.\begin{aligned} y_1 &= u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}}, \\ y_2 &= wu^{\frac{1}{3}} - \frac{H}{wu^{\frac{1}{3}}}, \\ y_3 &= w^2u^{\frac{1}{3}} - \frac{H}{w^2u^{\frac{1}{3}}}, \end{aligned}\right\} \tag{13}$$

where $u^{\frac{1}{3}}$ is any one of the three cube roots of u, and w is a complex cube root of unity (Art. 85).

Exercise. Test the solution by substitution of these values of y in (6).

By means of (2) and (13), the roots of equation (1) are *

$$\left.\begin{aligned} x_1 &= u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}} - \frac{a_1}{3\,a_0}, \\ x_2 &= wu^{\frac{1}{3}} - \frac{H}{wu^{\frac{1}{3}}} - \frac{a_1}{3\,a_0}, \\ x_3 &= w^2u^{\frac{1}{3}} - \frac{H}{w^2u^{\frac{1}{3}}} - \frac{a_1}{3\,a_0}. \end{aligned}\right\} \tag{14}$$

When the coefficients of the equation are real numbers, the numerical character of the roots depends upon the number under the radical sign in (11) and (12).

When $G^2 + 4\,H^3$ is negative, u is a complex number. In this case, to obtain y from (7) would involve the extraction of the cube root of complex numbers. As we have no general algebraic rule for extracting such a cube root, the case in which $G^2 + 4\,H^3$ is negative is called the **irreducible** case. These roots may, however, be obtained by a method involving trigonometry (see Art. 85). Even when $G^2 + 4\,H^3$ is positive, the solution presented above is not, in general, so well adapted to obtaining real roots of numerical equations as the methods of Arts. 102–105.

* This solution of the cubic is due to Tartaglia, but was first published by Cardan (1545).

108. The biquadratic equation. The general biquadratic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

may be written in the p-form (Art. 101) as

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0. \tag{1}$$

Adding $(mx + b)^2$ to both members of (1), we have

$$x^4 + p_1x^3 + (p_2 + m^2)x^2 + (p_3 + 2\,mb)x + p_4 + b^2 = (mx + b)^2. \tag{2}$$

Assume the identity,

$$x^4 + p_1x^3 + (p_2 + m^2)x^2 + (p_3 + 2\,mb)x + p_4 + b^2 \equiv (x^2 + \frac{p_1}{2}x + q)^2. \tag{3}$$

Equating coefficients of like powers of x, we have

$$p_2 + m^2 = \frac{p_1^2}{4} + 2\,q, \tag{4}$$

$$p_3 + 2\,mb = p_1q, \tag{5}$$

$$p_4 + b^2 = q^2. \tag{6}$$

Eliminating m and b from (4), (5), and (6), we obtain

$$(p_1^2 + 8\,q - 4\,p_2)(q^2 - p_4) = (p_1q - p_3)^2, \tag{7}$$

or $$8\,q^3 - 4\,p_2q^2 + (2\,p_1p_3 - 8\,p_4)q + 4\,p_2p_4 - p_1^2p_4 - p_3^2 = 0. \tag{8}$$

This is a cubic in q. Since the general cubic is solved by radicals in Art. 107, we may assume a value of q known. When q is known, the values of m and b are obtained from (4) and (6). From (2) and (3), we have

$$(x^2 + \frac{p_1}{2}x + q)^2 = (mx + b)^2, \tag{9}$$

which is equivalent to the two quadratic equations

$$x^2 + \frac{p_1}{2}x + q - mx - b = 0,$$

and

$$x^2 + \frac{p_1}{2}x + q + mx + b = 0.$$

The solutions of these two quadratics give the four roots of (1).*

* This solution is due to Ferrari and was first published by Cardan (1545).

EXERCISES

1. By Tartaglia's method, solve $x^3 - 4x^2 + 6x - 4 = 0$ and verify the results by substitution.

SOLUTION: Here $a_0 = 1$, $a_1 = -4$, $a_2 = 6$, $a_3 = -4$.

From (4) and (5), Art. 107,

$$G = -\tfrac{20}{27}, \quad H = \tfrac{2}{3}.$$

From (11),
$$u = \frac{10 + 6\sqrt{3}}{27},$$

$$u^{\frac{1}{3}} = \frac{1 + \sqrt{3}}{3}.$$

From (14) the roots of the given equation are

$$2,\ 1 + i,\ 1 - i.$$

Substitution for x shows that each of these numbers satisfies the equation to be solved.

2. Solve $x^4 - 6x^3 + 12x^2 - 20x - 12 = 0$. (1)

SOLUTION: Adding $(mx + b)^2$ to both members of this equation gives

$$x^4 - 6x^3 + (12 + m^2)x^2 + (2mb - 20)x + b^2 - 12 = (mx + b)^2. \quad (2)$$

Assume the identity

$$x^4 - 6x^3 + (12 + m^2)x^2 + (2mb - 20)x + b^2 - 12 = (x^2 - 3x + q)^2. \quad (3)$$

Equating coefficients, we obtain
$$12 + m^2 = 9 + 2q, \quad (4)$$
$$2mb - 20 = -6q, \quad (5)$$
$$b^2 - 12 = q^2. \quad (6)$$

Eliminating m and b from these three relations, we have the cubic

$$q^3 - 6q^2 + 42q - 68 = 0. \quad (7)$$

This cubic has a root $q = 2$. From (4), (5), and (6), the corresponding values of m^2, b^2, and mb are

$$m^2 = 1,\ b^2 = 16,\ mb = 4. \quad (8)$$

From (2), (3), and (8), $(x^2 - 3x + 2)^2 = (x + 4)^2$. (9)

This equation is equivalent to the two quadratic equations

$$x^2 - 3x + 2 - (x + 4) = 0, \quad (10)$$

and
$$x^2 - 3x + 2 + x + 4 = 0. \quad (11)$$

The roots of (10) are $2 \pm \sqrt{6}$, and those of (11) are $1 \pm i\sqrt{5}$. These four values satisfy the given biquadratic.

Solve the following equations, and verify the results by substitution.

3. $x^3 + 4x^2 + 4x + 3 = 0$,

4. $2x^3 - 3x^2 + 2x - 3 = 0$.

5. $3x^3 - 2x^2 - 6x + 4 = 0$.

6. $8x^3 - 28x^2 + 30x - 9 = 0$.

7. $x^4 + x^3 - x^2 - 7x - 6 = 0$.

8. $x^3 - 2x^2 + 3 = 0$.

9. $x^3 + 12x^2 + 57x + 74 = 0$.

10. $4x^4 - 36x^3 + 45x^2 + 54x - 81 = 0$.

109. Coefficients in terms of roots. Let $r_1, r_2, \cdots, r_n$ be the roots of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0, \tag{1}$$

Then, from Art. 94,

$$\begin{aligned} x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n &= (x - r_1)(x - r_2)\cdots(x - r_n), \\ &= x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + (r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)x^{n-2} \\ &\quad - (r_1r_2r_3 + \cdots + r_{n-2}r_{n-1}r_n)x^{n-3} + \cdots + (-1)^n r_1r_2r_3\cdots r_n, \end{aligned} \tag{2}$$

. . . . . .

by actual multiplication of the binomial factors of the second member.

Equating coefficients in (2) (Art. 94, Cor. I), we have

$$\left.\begin{aligned} -p_1 &= r_1 + r_2 + \cdots + r_n, \\ p_2 &= r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n, \\ -p_3 &= r_1r_2r_3 + \cdots + r_{n-2}r_{n-1}r_n, \\ &\cdots\cdots\cdots \\ (-1)^n p_n &= r_1r_2r_3\cdots r_n. \end{aligned}\right\} \tag{A}$$

That is, $-p_1 =$ sum of the roots,

$p_2 =$ sum of products of roots taken two at a time,

$-p_3 =$ sum of products of roots taken three at a time,

. . . . . .

$(-1)^n p_n =$ product of the roots.

If certain relations among the roots are given, the expressions (*A*) of the coefficients in terms of the roots may aid in solving the equation.

110. Variable coefficients and roots. Given that $r_1, r_2, \cdots, r_n$ are the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \tag{1}$$

relations (*A*), Art. 109, may be written in the form

$$\left.\begin{aligned} -\frac{a_1}{a_0} &= r_1 + r_2 + \cdots + r_n, \\ \frac{a_2}{a_0} &= r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n, \\ -\frac{a_3}{a_0} &= r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n, \\ &\cdots\cdots\cdots \\ (-1)^n\frac{a_n}{a_0} &= r_1r_2r_3\cdots r_n. \end{aligned}\right\} \tag{B}$$

If a_0 remains fixed, it follows from (B) that

$a_n \to^* 0$, if one root approaches zero;

$a_n \to 0$ and $a_{n-1} \to 0$, if two roots approach zero;

. . . . . .

$a_n \to 0$, $a_{n-1} \to 0$, $\cdots a_{n-r+1} \to 0$, if r roots approach zero.

In certain problems of analytic geometry, it is desirable to know the character of the coefficients of (1), if some of the roots become indefinitely large.

In (1), put $x = \frac{1}{x_1}$. This gives

$$\frac{a_0}{x_1^n} + \frac{a_1}{x_1^{n-1}} + \frac{a_2}{x_1^{n-2}} + \cdots + \frac{a_{n-1}}{x_1} + a_n = 0,$$

or, $$a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} + a_n x_1^n = 0. \qquad (2)$$

For our purposes, we may define $x = \infty$ (read, x becomes infinite) as $\frac{1}{x_1}$ when $x_1 \to 0$. The conditions which make $x = \infty$ are then precisely the conditions which make $x_1 \to 0$. Hence, from (2), if a_n remains fixed, it follows that

if $a_0 \to 0$, one root of equation (1) becomes infinite;

if $a_0 \to 0$, and $a_1 \to 0$, two roots of equation (1) become infinite;

. . . . . .

if $a_0 \to 0$, $a_1 \to 0$, $\cdots a_{r-1} \to 0$, r roots of equation (1) become infinite.

EXERCISES

1. Solve $x^3 - 2x^2 - 4x + 8 = 0$, the sum of two of the roots being 2.

2. Solve $2x^3 - 3x^2 + 2x - 3 = 0$, the sum of two of the roots being zero.

3. The roots of $x^3 - 6x^2 + 3x + 10$ are in arithmetical progression. Find them.

4. Solve $x^3 - 8x^2 + 5x + 50 = 0$, two of the roots being equal.

5. Obtain the roots of $ax^2 - 13x + 1 = 0$, to three decimal places, when $a = 10, 1, 0.1, 0.01$, and 0.001.

6. Obtain the roots of $x^2 - 13x + a = 0$, to three decimal places, when $a = 10, 1, 0.1, 0.01, 0.001$.

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* The symbol $\to$ is read "approaches."

7. Determine m and b so that the quadratic

$$3x^2 - (mx + b)^2 - 4x + 2 = 0$$

shall have two infinite roots.

8. Determine m and b so that the cubic

$$x^3 + x(mx + b)^2 - 3x^2 + 5 = 0$$

shall have two infinite roots.

9. Determine m and b so that the quadratic

$$9x^2 - 16(mx + b)^2 = 25$$

shall have two infinite roots.

10. Determine m and b so that

$$3x^2 + 34x(mx + b) + 11(mx + b)^2 - x + 21(mx + b) = 0$$

shall have two infinite roots.

CHAPTER XIV

LOGARITHMS

111. Generalization of exponents. In Art. 6, a^x is defined when x is a positive integer; and a meaning is obtained (Arts. 7–9) from the laws of exponents for a^x when x is any rational number. Thus, $4^5 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$, and $8^{\frac{2}{3}}$ is the square of the cube root of 8. But no meaning has been obtained thus far for a^x when x is irrational; for example, $4^{\sqrt{2}}$ is thus far undefined. We have, however, defined $\sqrt{2}$ as the limit of a sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \cdots.$$

When a variable, z, taking this sequence of rational values, approaches $\sqrt{2}$ as a limit, it can be proved that $a^z (a < 0)$ has a limit, and we define $a^{\sqrt{2}}$ as this limit. In accordance with this illustration, $a^z (a > 0)$, if x is irrational, is defined as the limit of a^z when z approaches x.*

112. Definition of a logarithm. *If* $a^x = y (a > 0,\ a \neq 1)$, *then* x *is said to be the* **logarithm** *of* y *to the base* a, and this is written $x = \log_a y$.

The two equations $a^x = y$ (1)

and $x = \log_a y$ (2)

thus mean exactly the same thing; and the terms logarithm and exponent are equivalent.

We shall assume in what follows that:

1. *Corresponding to any two positive numbers* y *and* a $(a \neq 1)$ *there exists one and only one real number* x *such that* $a^x = y$.

This assumption is sometimes expressed by saying that any

* a^x can also be defined, consistently with the laws of exponents, as the limiting value of an infinite series of positive integral powers of x (Art. 170).

positive number has one and only one logarithm, whatever positive number is the base (unity excepted).

2. *The laws of exponents (Art. 6) which apply to rational exponents are also valid when irrational exponents are involved.*

EXERCISES

1. $\log_2 8 = ?$ $\log_a a = ?$ $\log_4 2 = ?$ $\log_4 256 = ?$ $\log_6 4 = ?$

2. Fill out the following table.

BASE	NUMBER	LOGARITHM
	81	4
10		5
	23	$\frac{1}{3}$
2	$\frac{1}{32}$	
3	$\frac{1}{27}$	
	32	-5

113. Derived properties of logarithms.

1. *The logarithm of a product equals the sum of the logarithms of its factors.*

Let $\log_a u = x$ and $\log_a v = y$, (1)

then, $a^x = u$, $a^y = v$, (Definition of logarithm.)

and $uv = a^{x+y}$. (Art. 6 and Art. 112, Assumption 2.)

Hence, $\log_a uv = x + y$,

that is, $\log_a uv = \log_a u + \log_a v$.

Similarly, $\log_a(uvw) = \log_a u + \log_a v + \log_a w$,

and so on for any number of factors.

EXAMPLE: $\log_{10} 255 = \log_{10} 3 + \log_{10} 5 + \log_{10} 17$.

2. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

As above, let $\log_a u = x$ and $\log_a v = y$,

then, $a^x = u,\ a^y = v,$

and $\dfrac{u}{v} = a^{x-y}.$

Hence, $\log_a \dfrac{u}{v} = x - y,$

that is, $\boldsymbol{\log_a \dfrac{u}{v} = \log_a u - \log_a v.}$

EXAMPLE: $\log_{10} \frac{6}{1}\frac{2}{3}\frac{5}{3} = \log_{10} 625 - \log_{10} 133.$

3. *The logarithm of u^v is equal to v times the logarithm of u.*

To prove this, let $x = \log_a u$ or $a^x = u.$ (1)

Then, from (1), $u^v = a^{vx}.$

(Law of Indices, and Art. 112, 2.)

Hence, $\boldsymbol{\log_a u^v = vx = v \log_a u.}$ (2)

EXAMPLE: $\log_{10} (257)^{\frac{1}{2}} = \frac{1}{2} \log_{10} 257.$

Making $v = n$ and $v = \dfrac{1}{n}$ respectively, we have

(*a*) *The logarithm of the nth power of a number is n times the logarithm of the number.*

(*b*) *The logarithm of the real positive nth root of a number is the logarithm of the number divided by n.*

EXERCISES

Express the logarithms of the following expressions in terms of the logarithms of integers.

1. * $\log \dfrac{\sqrt[4]{8}}{9^{\frac{1}{5}} 6^{\frac{2}{3}}}.$

SOLUTION: $\log \dfrac{\sqrt[4]{8}}{9^{\frac{1}{5}} 6^{\frac{2}{3}}} = \log \sqrt[4]{8} - \log 9^{\frac{1}{5}} - \log 6^{\frac{2}{3}}$ (1 and 2, Art. 113.)

$= \frac{1}{4} \log 8 - \frac{1}{5} \log 9 - \frac{2}{3} \log 6.$ (3, Art. 113.)

2. $\log \dfrac{2^2}{3^3}.$ **3.** $\log \dfrac{\sqrt{13}}{\sqrt[5]{10}\sqrt[3]{48}}.$ **4.** $\log \dfrac{25^{\frac{1}{3}}}{11^{\frac{2}{3}} 23^{\frac{3}{4}}}.$

* When in a problem the same base is used throughout, it is customary not to write the base.

Express the logarithms of the following in terms of the logarithms of prime numbers.

5. $\log \dfrac{(35)^{\frac{1}{2}}}{(36)^2(72)^{\frac{1}{4}}}$.

6. $\log \dfrac{(88)^{-\frac{1}{2}}}{(75)^{\frac{2}{4}}(12)^2}$.

7. $\log \dfrac{(100)^2}{(20)^{\frac{1}{2}}(75)^{\frac{2}{3}}}$.

8. $\log (\sqrt{2}\sqrt[3]{7^2}\sqrt[5]{6})$.

9. $\log (\sqrt[3]{2}\sqrt{66}\sqrt[5]{25})$.

10. Prove that $\log_a 1 = 0$.

Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, find the logarithms of the following numbers to the base 10.

11. 12. **12.** 30. **13.** 42. **14.** 420.

15. 189. **16.** 900. **17.** 343. **18.** $\frac{343}{16}$.

19. $\frac{35}{48}$. **20.** $\frac{1}{2^5 3^2}$. **21.** $\frac{1}{10^2 3^3}$. **22.** $\sqrt{504}$.

23. $\sqrt[3]{294}$. **24.** $\sqrt{\frac{1}{2}}$. **25.** $\sqrt[5]{1715}$. **26.** $\sqrt[6]{43218}$.

114. Common logarithms. While any positive number can be used as the base of some system of logarithms, there are two systems in general use. These are the *common* or Briggs's system and the *natural* or Naperian system. In the *common system* the base is 10, while in the *natural* system the base is a certain irrational number $e = 2.71828 \cdots$. It may be stated that the common system is adapted to numerical computation, while the natural system is adapted to analytical work.*

In the following discussion of common logarithms, $\log x$ is written as an abbreviation of $\log_{10} x$.

Since,

$10^0 = 1$ $\quad$ $10^{-1} = 0.1$
$10^1 = 10$ $\quad$ $10^{-2} = 0.01$
$10^2 = 100$ $\quad$ $10^{-3} = 0.001$
$10^3 = 1000$ $\quad$ $10^{-4} = 0.0001$
. . . $\quad$ . . . .

it follows that

$\log 1 = 0$ $\quad$ $\log 0.1 = -1$
$\log 10 = 1$ $\quad$ $\log 0.01 = -2$
$\log 100 = 2$ $\quad$ $\log 0.001 = -3$
$\log 1000 = 3$ $\quad$ $\log 0.0001 = -4$
. . . . $\quad$ . . . . .

* The notation $\ln x$ for $\log_e x$ and $\log x$ for $\log_{10} x$ is frequently used when both kinds of logarithms appear in the same problem.

So far as these powers of 10 are concerned, it may be observed that the logarithm of the number becomes greater as the number increases. In accordance with this observation, we may assume, if $a < x < b$, that

$$\log a < \log x < \log b. \tag{1}$$

For example, $\log 100 < \log 765 < \log 1000,$

or $2 < \log 765 < 3.$

When the logarithm of a number is not an integer, it may be represented at least approximately by means of decimal fractions. Thus, $\log 765 = 2.8837$ correct to four decimal places.

The integral part of a logarithm is called the **characteristic** and the decimal part is called the **mantissa**. In $\log 765$, the characteristic is 2 and the mantissa is 0.8837. For convenience in constructing tables, it is desirable to select the mantissa as positive even if the logarithm is a negative number. For example, $\log \frac{1}{2} = -0.3010$; but since $-0.3010 = 9.6990 - 10$, this may be written $\log \frac{1}{2} = 9.6990 - 10$ with a positive mantissa. The following illustration shows the method of writing the characteristic and mantissa:

$$\begin{aligned}
\log 7185 &= 3.8564 \\
\log 718.5 &= 2.8564 \\
\log 71.85 &= 1.8564 \\
\log 7.185 &= 0.8564 \\
\log 0.7185 &= 9.8564 - 10 \\
\log 0.07185 &= 8.8564 - 10
\end{aligned}$$

115. Characteristic. With our decimal system of notation, the characteristic in the case of the base 10 is very easy to determine by a simple rule. Herein lies the advantage of this base.

If y is a number which has n digits in the integral part, then

$$10^{n-1} \leqq y < 10^n, \tag{1}$$

and by Art. 114, (1), $n - 1 \leqq \log y < n.$

Hence, $\log y = n - 1 + f,$

where f is positive and less than 1.

Hence, *to find the characteristic of the common logarithm of a number which has an integral part, subtract* 1 *from the number of digits in the integral part.*

If y represents a decimal fraction, we may move the decimal point ten places to the right, and apply the rule just stated to the integral part of the number so formed, provided we subtract 10 from the resulting logarithm. That is,

$$y = \frac{10^{10}\, y}{10^{10}} = \frac{y_1}{10^{10}},$$

where

$$\begin{aligned} \log y &= \log y_1 - \log 10^{10}, \\ &= \log y_1 - 10. \end{aligned}$$

The result so obtained could manifestly also be obtained by the following rule:

To find the characteristic of the common logarithm of a decimal fraction, subtract from 9 *the number of ciphers between the decimal point and the first significant figure. From the number so obtained subtract* 10.

If two numbers contain the same sequence of figures, and therefore differ only in the position of the decimal point, the one number is the product of the other and an integral power of 10, and hence, by Art. 113, the logarithms of the numbers differ only by an integer. Thus,

$$\begin{aligned} \log 3722 &= \log 37.22 + \log 100 \\ &= \log 37.22 + 2. \end{aligned}$$

Hence, *the mantissa of the common logarithm of a number is independent of the position of the decimal point.* In other words, the common logarithms of two numbers which contain the same sequence of figures differ only in their characteristics. Hence, tables of logarithms contain only the mantissas, and the computer must find the characteristics by the foregoing rules.

116. Use of tables. On pp. 162, 163, a "four-place" table of logarithms is given. In this table, the mantissas of the logarithms of all integers from 1 to 999 are recorded correct to four decimal places. "Five-place," "six-place," and "seven-place" tables are in common use, but this four-place table will serve for our present purposes.

Methods by which such a table can be made will be discussed after applying the logarithms found in the table to purposes of

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

arithmetical calculation. In order to use the tables we must know how to take from the tables the logarithm of a given number, and how to take from the tables the number which has a given logarithm.

117. To find from the table the logarithm of a given number.

EXAMPLES

1. Find the logarithm of 821.

Glance down the column headed N for the first two significant figures, then at the top of the table for the third figure. In the row with 82 and the column with 1 is found 9143.

Hence, $\log 821 = 2.9143$.

2. Find the logarithm of 68.42.

This number has more than three significant figures, so that its logarithm is not recorded in the table. It may, however, be obtained approximately from logarithms recorded in the table by a process of **interpolation.** In this process, it is assumed that to a small change in the number, there corresponds a change in the logarithm which is proportional to the change in the number. This assumption is called the principle of proportional parts. As in Ex. 1, we find that the mantissas of 6840 and 6850 are 8351 and 8357, respectively. The difference between these two mantissas is 6. Since 6842 is two tenths of the interval from 6840 to 6850, by the principle of proportional parts, we add to 8351, $0.2 \times 6 = 1^{+}$.

Hence, $\log 68.42 = 1.8352$.

118. To find from the table the number which corresponds to a given logarithm.

EXAMPLES

1. Find the number whose logarithm is 2.4675. The mantissa 4675 is not recorded in the table, but it lies between the two adjacent mantissas 4669 and 4683 of the table. The mantissa 4669 corresponds to the number 293 and 4683 corresponds to 294. The number 4675 is $\frac{6}{14}$ of the interval from 4669 to 4683. By the principle of proportional parts, the number whose mantissa is 4675 is $2930 + \frac{6}{14} \times 10 = 2934^{+}$.

Hence, $\log 293.4 = 2.4675$.

2. Find the number whose logarithm is $9.3025 - 10$.

From the table, $\log 0.2000 = 9.3010 - 10$

$\log 0.2010 = 9.3032 - 10$

Difference $= 0.0022$

$(9.3025 - 10) - (9.3010 - 10) = 0.0015$.

By the principle of proportional parts, the number is

$0.2000 + \frac{15}{22} \times 0.0010 = 0.2007$.

EXERCISES

Obtain, from the table, the common logarithms of the following.

1. 163.	**2.** 89.	**3.** 999.
4. 1.41.	**5.** 0.00785.	**6.** 6563.
7. 7.854.	**8.** 3.142.	**9.** 0.5236.
10. 1.732.	**11.** 0.8665.	**12.** 0.0298.

Obtain, by means of the table, the numbers whose common logarithms are the following.

13. 2.7182.	**14.** $9.8532 - 10$.	**15.** 3.1416.
16. 0.5236.	**17.** $7.8321 - 10$.	**18.** 4.2631.
19. $8.5432 - 10$.	**20.** 1.4142.	**21.** 0.4343.

119. Computation by means of logarithms. The application of logarithms to shorten calculations depends upon the properties of logarithms given in Art. 113. By means of logarithms laborious multiplications and divisions may be replaced by additions and subtractions; and involution and evolution may be replaced by multiplication and division.

EXAMPLES

1. Find the value of $N = \dfrac{6.320 \times 8.674}{2.851}$ to four significant figures.

$$\log 6.320 = 0.8007$$
$$\log 8.674 = 0.9382$$
$$\log (6.320)(8.674) = \overline{1.7389}$$
$$\log 2.851 = 9.4550$$
$$\log N = \overline{1.2839}$$
$$N = 19.23.$$

In using logarithms, *much time is saved and the liability of error is decreased by making a so-called form for all the work before using the table at all.*

Thus, in Example 1, the "form" is

$$\log 6.320 =$$
$$\log 8.674 =$$
$$\log (6.320)(8.674) =$$
$$\log 2.851 =$$
$$\log N =$$
$$N =$$

2. Make a form for evaluating $N = \dfrac{(6.85)^{\frac{1}{2}}\sqrt[3]{8.542}}{\sqrt{65.27}}$.

$$\log 6.85 =$$
$$\log (6.85)^{\frac{1}{2}} =$$
$$\log 8.542 =$$
$$\log \sqrt[3]{8.542} =$$
$$\log (6.85)^{\frac{1}{2}}\sqrt[3]{8.542} =$$
$$\log 65.27 =$$
$$\log \sqrt{65.27} =$$
$$\log N =$$
$$N =$$

3. Evaluate $N = \sqrt[3]{-58.61}$.*

$$\log 58.61 = 1.7680\,n$$
$$\log \sqrt[3]{58.61} = 0.5893\,n$$
$$N = -3.885.$$

EXERCISES AND PROBLEMS

Evaluate to four significant figures by logarithms.

1. $(0.2386)^3$.

2. $0.0631 \times 7.208 \times 0.5127$.

3. $\dfrac{0.3384}{8.659}$.

4. $\dfrac{2563 \times (-3.442)}{714.8 \times 0.5110}$.

5. $\sqrt[3]{645.3}$.

6. $(0.9323)^5$.

7. $\dfrac{763.2 \times 2.163}{986.7}$.

8. $\sqrt[3]{0.08244}$.

9. $\sqrt[5]{-0.62305}$.

10. $\dfrac{\sqrt[3]{0.0001289}}{\sqrt[4]{0.0008276}}$.

11. $\dfrac{\sqrt[3]{8.193} \times \sqrt{(0.06285)^3}}{0.9834}$.

12. $\sqrt[3]{2} \times \sqrt[3]{10} \times \sqrt[7]{0.01}$.

13. $\sqrt[7]{0.7684}$.

14. $(0.008543)^{\frac{1}{5}}$.

15. $(0.4754)^{\frac{3}{4}}(0.6782)^{\frac{2}{3}}$.

16. $\dfrac{\sqrt{4300}}{(1.06)^{\frac{1}{3}}}$.

17. $\sqrt{\dfrac{(3.142)(0.5236)}{(85^{\frac{2}{3}})}}$.

18. $\sqrt[3]{\dfrac{-0.03296}{7.962}}$.

19. $\sqrt[5]{3.1416 \times (16)^3}$.

20. $\sqrt[3]{185^2 - 112^2}$.

HINT: $185^2 - 112^2 = (185 + 112)(185 - 112)$.

* When a number is negative, find its logarithm without regard to sign, writing n after a logarithm that corresponds to a negative number so as to keep the negative sign in mind.

21. $\sqrt[5]{210^2 - 167^2}$.

22. $\sqrt[4]{100}$.

23. $(-0.03574)^{\frac{4}{5}}$.

24. $(\frac{7}{8})^{\frac{3}{4}}$.

25. $\dfrac{12(0.5236^{\frac{3}{5}})}{(-52.36)^{\frac{5}{9}}}$.

26. $\dfrac{(\sqrt{5})(\sqrt[3]{7})}{(\sqrt[4]{8})(\sqrt[5]{9})}$.

27. The time t of oscillation of a simple pendulum of length l feet is given in seconds by the formula

$$t = \pi\sqrt{\frac{l}{32.16}}.$$

Find the time of oscillation of a pendulum 3.326 feet long. (Take $\pi = 3.142$.)

28. What is the weight in tons of a solid cast-iron sphere whose radius is 5.343 feet, if the weight of a cubic foot of water is 62.355 pounds and the specific gravity of cast-iron is 7.154 ?

29. Find the volume and surface of a sphere of radius 14.71.

30. The stretch of a brass wire when a weight is hung at its free end is given by the relation

$$S = \frac{mgl}{\pi r^2 k},$$

where m is the weight applied, $g = 980$, l is the length of the wire, r is its radius, and k is a constant. Find k for the following values: $m = 944.2$ grams, $l = 219.2$ centimeters, $r = 0.32$ centimeter, and $S = 0.060$ centimeter.

31. Find the length l of a wire which stretches 5.9 centimeters for a weight of 1826.5 grams hanging at its free end, the diameter of the wire being 0.064 centimeter, and $k = 1.1 \times 10^{12}$.

32. The weight P in pounds which will crush a solid cylindrical cast-iron column is given by the formula

$$P = 98{,}920\,\frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches and l the length in feet. What weight will crush a cast-iron column 6 feet long and 4.3 inches in diameter ?

33. For wrought-iron columns the crushing weight is given by

$$P = 299{,}600\,\frac{d^{3.55}}{l^2}.$$

What weight will crush a wrought-iron column of the same dimensions as that in Problem 32 ?

34. The weight W of one cubic foot of saturated steam depends upon the pressure in the boiler according to the formula

$$W = \frac{P^{0.941}}{330.36},$$

where P is the pressure in pounds per square inch. What is W if the pressure is 280 pounds per square inch ?

35. By using the formula given in Problem 34, find the pressure in a boiler when a cubic foot of steam weighs just one pound.

36. The diameter in inches of a connecting rod depends upon the diameter D of the engine cylinder, l the length of the connecting rod, and P the maximum steam pressure in pounds per square inch, according to Mark's formula

$$d = 0.02758\sqrt{D \cdot l \cdot \sqrt{P}}.$$

What is d when $D = 30$, $l = 75$, and $P = 150$?

37. For $D = 10$, $l = 60$, a table in Kent's *Pocket Book*, based upon the formula in Problem 36, gives $d = 2.14$ inches. What was the maximum steam pressure used?

38. The discharge of water from a triangular weir is given by

$$q = \frac{8\,c}{15} H^{\frac{5}{2}} \sqrt{2\,g},$$

where c is a constant 0.592, g is the acceleration due to gravity 32.2 feet per second, and H is the water head. Find q when $H = 0.3$ foot.

39. Given $pv^{1.41} = 400$ as the relation between pressure and volume of air expanding under certain conditions. Find p when $v = 24$.

40. The number, n, of vibrations per second made by a stretched string is given by the relation

$$n = \frac{1}{2\,l}\sqrt{\frac{Mg}{m}},$$

where l is the length of the string, M the weight used to stretch the string, m the weight of one centimeter of the string, and $g = 980$. Find n, when $M = 6213.6$ grams, $l = 84.9$ centimeters, and $m = 0.00670$ gram.

41. What must be the weight per centimeter length of a wire which is 70.95 centimeters long and is stretched by a weight of 4406.5 grams, in order that it may vibrate 178 times per second?

42. The work in foot pounds done during the adiabatic expansion of a gas from pressure p_1 to pressure p_2 is

$$W = 144\,\frac{p_1 v_1}{k-1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]$$

where v_1 is the original volume of the gas and k is a constant. Find W for $k = 1.41$, $p_1 = 60$, $p_2 = 15$, $v_1 = 3.5$.

43. If \$1500 is placed at 3 per cent interest, converted annually, what will it amount to in 10 years?

HINT: In n years \$1 will amount to $\$\,(1.03)^n$.

44. What will \$10,000 amount to if left at interest for 10 years at 4 per cent: (*a*) converted annually? (*b*) converted semiannually? (*c*) converted quarterly?

45. What sum of money left for 20 years at 5 per cent, converted annually, will amount to \$10,000 at the end of that time?

46. If \$1 had been kept at interest 5 per cent, compounded annually, from the beginning of the Christian era to the present time, how many digits would occur in the integral part of the accumulated amount when expressed in dollars?

47. The formula $y = 0.0263\,x^{1.1}$ gives the relation between y and x when x stands for the stress in kilograms per square centimeter of cross section of a hollow cast-iron tube subject to tensile stress, and y for the elongation of the tube in terms of $\frac{1}{600}$ cm. as a unit. Compute y when $x = 101.8$.

48. The formula $y = ks^x g^{cx}$, where $\log k = 5.03370116$, $\log s = -0.003296862$, $\log g = -0.00013205$, $\log c = 0.04579609$, gives the number living at age x in Hunter's Makehamized American Experience Table of Mortality. Find, to such a degree of accuracy as you can secure with a four-place table of logarithms, the number living (1) at age 10, (2) at age 30.

49. Given that one kilometer is equal to 0.6214 mile. Find the number of miles in 2489 kilometers.

50. Given that one kilometer equals 0.6214 mile, and that the area of Illinois is 56,625 square miles. Express the area of Illinois in square kilometers (to four significant figures).

120. Change of base. *The logarithm of a number y to the base b is equal to the product of its logarithm to the base a and the logarithm of a to the base b.*

That is, $$\log_b y = \log_a y \cdot \log_b a. \tag{1}$$

Let $$u = \log_a y \text{ and } v = \log_b y. \tag{2}$$

Then, $$a^u = y,\ b^v = y, \tag{3}$$

and $$a^u = b^v. \tag{4}$$

$$a = b^{\frac{v}{u}}, \tag{5}$$

$$\frac{v}{u} = \log_b a,$$

$$v = u \log_b a. \tag{6}$$

From (2) and (6), $$\log_b y = \log_a y \log_b a. \tag{7}$$

EXAMPLE: $$\log_{10} 128 = \log_2 128 \log_{10} 2.$$

By making $y = b$ in (7), we obtain

$$1 = \log_a b \log_b a.$$

That is, $$\log_b a = \frac{1}{\log_a b}. \tag{8}$$

The number $\log_b a$ is often called the **modulus** of the system of base b with respect to the system of base a.

In Art. 115, attention is called to the advantages of 10 for the base of a system of logarithms to be used in numerical calculations. For analytical purposes, as will appear in the calculus, it is convenient to use *natural* logarithms. This system has for its base an irrational number $e = 2.71828\cdots$. In the chapter on Infinite Series, there will be given a series from which this approximation to e is obtained, and another series from which the logarithm of a number to the base e can be obtained to any number of decimal places. It turns out that

$$\log_e 10 = 2.3026,$$

and

$$\log_{10} e = \frac{1}{\log_e 10} = 0.4343.$$

By (1),

$$\log_{10} y = \log_e y \log_{10} e,$$
$$= 0.4343 \log_e y.$$

The number $\log_{10} e = 0.4343$ is the **modulus** (to four significant figures) of common logarithms with respect to natural logarithms.

EXERCISES

1. Given $\log_e 2 = 0.6931$, find $\log_{10} 2$ and compare the result with the value in table, p. 162.

2. Given $\log_{10} 2 = 0.3010$, find $\log_e 4$.

3. Given $\log_{10} 3$, find $\log_5 3$.

HINT: By Art. 120,
$$\log_5 3 = \log_{10} 3 \cdot \log_5 10,$$
$$= \frac{\log_{10} 3}{\log_{10} 5}.$$

4. Given $\log_{10} 3$, find $\log_5 81$.

Find the logarithms of the following numbers.

5. 10 to the base 2.

6. 10 to the base 4.

7. 10 to the base 3.

8. 10 to the base 5.

9. 75 to the base 3.

10. 13 to the base 20.

11. 130 to the base 20.

12. 1300 to the base 20.

121. Graph of $y = \log_a x (a > 1)$. A general notion of the value of the logarithm of any number can be easily fixed by reference to the graph of $y = \log_a x$. This graph is also the graph of $x = a^y$. In the graph (Fig. 32) we take $a = e = 2.718\cdots$, but

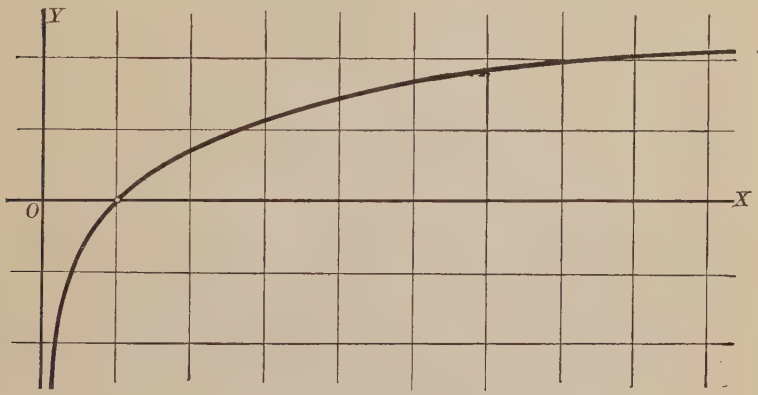

FIG. 32.

the general form of the curve is not changed if a be given any other positive value greater than 1. If the student retains this picture, he should find it easy to keep in mind the following facts when the base is greater than unity.

1. A negative number does not have a real number for its logarithm.
2. The logarithm of a positive number is positive or negative according as the number is greater than or less than 1.
3. If x approaches zero, $\log x$ decreases without limit.
4. If x increases indefinitely, $\log x$ increases without limit.

EXERCISES

1. Plot the graph of $y = \log_{10} x$ by using tables to find $\log_{10} x$.

2. Plot the graph of $y = \log_5 x$.

HINT: $$\log_5 x = \frac{\log_{10} x}{\log_{10} 5}.$$

3. Plot the graph of $x = \log_5 y$.

4. Plot the graph of $x = \log_2 y$.

122. Exponential and logarithmic equations. An equation which involves the unknown or unknowns in the exponents is often called an **exponential equation.** Thus, $2^x = 16$ is an exponential equation in x. In this simple example, the value of x can be obtained by inspection; but a table of logarithms is, in general, of value in solving exponential equations.

Such equations arise in a variety of problems. For instance, at compound interest, the amount of one dollar at a nominal rate of 0.05 per annum is (see Problem 43, p. 168)

$$A = \left(1 + \frac{0.05}{n}\right)^{nt} \text{dollars},$$

in which t is the time in years and n is the number of times per year that interest is converted. We may also write

$$A = \left[\left(1 + \frac{0.05}{n}\right)^{\frac{n}{0.05}}\right]^{0.05t}.$$

When n is increased beyond bound, the interest is said to be converted continuously. It turns out that, in this case (see Art. 170),

$$A = e^{0.05t},$$

where e is the base of natural logarithms.

EXAMPLE: What will \$1000 amount to in one year at 5% interest converted continuously?

SOLUTION: Let S be the amount of \$1000 at the end of a year, then

$$S = 1000\, e^{0.05},$$
$$\log S = \log 1000 + 0.05 \log e = 3.0217,$$
$$S = \$1051.$$

An equation which involves the logarithm of an expression that contains an unknown is sometimes called a **logarithmic** equation. Thus,

$$\log_{10} 2x = 3$$

is a logarithmic equation. To solve this equation, we may write, from the definition of a logarithm,

$$2x = 10^3 = 1000.$$

Hence,

$$x = 500.$$

EXERCISES AND PROBLEMS

Solve the following equations for x.

1. $5^x = 10$.

SOLUTION: Since $5^x = 10$,

$$\log_{10} 5^x = \log_{10} 10 = 1.$$

$$x \log_{10} 5 = 1.$$

$$x = \frac{1}{\log_{10} 5}$$

$$= \frac{1}{.6990} = 1.431.$$

2. $2^{3x}\, 5^{2x-1} = 4^{5x}\, 3^{x+1}$.

SOLUTION: $\log_{10} 2^{3x}\, 5^{2x-1} = \log_{10} 4^{5x}\, 3^{x+1}$,

$$3x \log_{10} 2 + (2x - 1) \log_{10} 5 = 5x \log_{10} 4 + (x + 1) \log_{10} 3$$

$$= 10x \log_{10} 2 + (x + 1) \log_{10} 3.$$

Transposing and collecting terms, we have

$$x(2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3) = \log_{10} 3 + \log_{10} 5.$$

$$x = \frac{\log_{10} 3 + \log_{10} 5}{2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3}$$

$$= \frac{0.4771 + 0.6990}{1.3980 - 2.1070 - 0.4771}$$

$$= -0.9916.$$

3. $16 = \log_{10} x^2$.

SOLUTION:

$$16 = \log_{10} x^2, \tag{1}$$

From (1),

$$x^2 = 10^{16}, \tag{2}$$

$$x = \pm 10^8. \tag{3}$$

4. $11^x = 7$.

5. $(0.3)^x = 0.8$.

6. $3^{x^2} = 532$.

7. $5^{(x^2-3x)} = \frac{1}{25}$.

8. $21^{x^2-2x} = 9261$.

9. In a geometrical progression, $l = ar^{n-1}$, solve for n in terms of a, l, and r.

10. In a geometrical progression, $s = \dfrac{ar^n - a}{r - 1}$, solve for n in terms of a, r, and s.

Solve for x and y the following systems of equations.

11.

$$5^{x+y} = 82, \tag{1}$$

$$3^{x-y} = 4. \tag{2}$$

SOLUTION: From (1) and (2),

$$(x + y) \log 5 = \log 82, \tag{3}$$

$$(x - y) \log 3 = \log 4. \tag{4}$$

Solving the linear equations (3) and (4) for x and y, we get

$$x = \frac{\log 82}{2 \log 5} + \frac{\log 4}{2 \log 3}, \tag{5}$$

$$y = \frac{\log 82}{2 \log 5} - \frac{\log 4}{2 \log 5}. \tag{6}$$

Complete by computing the value of (5) and (6) to three decimal places by the use of tables.

12. $2^{x+y} = 18$,
$3^x = 2^y$.

13. $4^{x+y} = 6^{2x}$,
$\log (x + 1) = \log (y - 3)$.

14. In how many years will \$1000 amount to \$2000 at a nominal rate of 0.06 per annum, (1) when interest is converted annually, (2) when it is converted quarterly, (3) when it is converted continuously?

15. Solve for x the equation $e^x + e^{-x} = y$; (a) when $y = 2$, (b) when $y = 4$.

16. If fluid friction be used to retard the motion of a flywheel making V_0 revolutions per minute, the formula $V = V_0 e^{-kt}$ gives the number of revolutions per minute, after the friction has been applied t seconds. If the constant $k = 0.35$, how long must the friction be applied to reduce the number of revolutions from 200 to 50 per minute?

17. The pressure, P, of the atmosphere in pounds per square inch, at a height of z feet, is given approximately by the relation

$$P = P_0 e^{-kz},$$

where P_0 is the pressure at sea level and k is a constant. Observations at sea level give $P_0 = 14.72$, and at a height of 1122 feet, $P = 14.11$. What is the value of k?

18. Assuming the law in Problem 17 to hold, at what height will the pressure be half as great as at sea level?

19. If a body of temperature $T_1°$ be surrounded by cooler air of temperature $T_0°$, the body will gradually become cooler and its temperature, $T°$, after a certain time, say t minutes, is given by Newton's law of cooling, that is,

$$T = T_0 + (T_1 - T_0) e^{-kt},$$

where k is a constant. In an experiment a body of temperature 55° C. was left to itself in air whose temperature was 15° C. After 11 minutes the temperature was found to be 25°. What is the value of k?

20. Assuming the value of k found in Problem 19, what time will elapse before the temperature of the body drops from 25° to 20°?

21. Solve the equation $\log_e (3x + 1) = 2$ for x.

22. Solve the equation $\log_{10} (x^2 - 21x) = 2$ for x.

23. In solving an important problem in the elements of mechanics, it turns out that

$$t = \frac{1}{k} \log_e \frac{ks + \sqrt{k^2 s^2 + v_0^2}}{v_0}, \tag{1}$$

where s is the distance traversed by a moving point in time t. It is, in general, more useful to have s in terms of t than t in terms of s. Hence, express s in terms of t in equation (1).

123. Calculation of logarithms. At this point the inquiring student will naturally bring up the question as to how the logarithms of numbers are computed so as to make a table of logarithms. Logarithms were invented by Napier about the year 1600 and common logarithms by Briggs a little later. The invention grew out of the comparison of two series of numbers — the one in arithmetical progression and the other in geometrical progression. The following theorem lies at the foundation of the early methods of computing logarithms:

If a series of numbers are in geometrical progression, their corresponding logarithms are in arithmetical progression.

Let the numbers in geometrical progression be

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}. \tag{1}$$

Then, $$\log a,\ \log ar,\ \log ar^2,\ \log ar^3,\ \cdots,\ \log ar^{n-1} \tag{2}$$

are in arithmetical progression.

In this arithmetical progression, the first term is $\log a$, and the common difference is $\log r$. The following example illustrates the use of this principle in calculating logarithms. Given $\log 1 = 0$, $\log_{10} 1000 = 3$, the geometrical mean between 1 and 1000 is $\sqrt{1000} = 31.62$. Then 1, 31.62, 1000 is a geometrical progression, and 0, 1.5, 3 is the corresponding arithmetical progression, so that $1.5 = \log_{10} 31.62$. Next, insert a geometric mean between 1 and 31.62, also between 31.62 and 1000. This gives

1, 5.624, 31.62, 177.8, 1000 as the geometrical series,

and 0, 0.75, 1.5, 2.25, 3 as the corresponding logarithms.

We could next insert between any two of these numbers a geometrical mean, and find its logarithm. By continuing this process, we could insert means until the numbers would differ by as little

as we please. This method of calculating logarithms has the disadvantage of giving the logarithms of numbers spaced unequally, since the numbers are in geometrical progression.

Another method of obtaining logarithms, which has many advantages over the one just given, is discussed briefly in Art. 171 of the chapter on Infinite Series; but its more complete treatment belongs to the calculus.

CHAPTER XV

PARTIAL FRACTIONS

124. Introduction. Early in the study of algebra we added together algebraic fractions and found the sum to be a single fraction whose denominator is the lowest common multiple of the denominators. Thus,

$$\frac{6}{x+1}+\frac{3}{x+2}=\frac{9x+15}{x^2+3x+2}.$$

It is often necessary to perform the inverse operation, that is, to decompose a given fraction into a sum of other fractions (called "partial" fractions) having denominators of lower degree. Thus it is easily shown that $\frac{2x}{x^2-1}$ can be decomposed into $\frac{1}{x+1}+\frac{1}{x-1}$.

An algebraic fraction is said to be *proper* when its numerator is of lower degree than its denominator. In this chapter it is necessary to consider only proper fractions; for if the degree of the numerator is not lower than that of the denominator, the fraction may be reduced by division to the sum of an integral part and a proper fraction. Thus,

$$\frac{3x^4-3x^2+2x}{x^2-1}=3x^2+\frac{2x}{x^2-1}.$$

We shall *assume* the possibility of decomposing any proper fraction whose denominator contains factors prime to each other into the partial fractions of the types

$$\frac{A}{x-a},\quad \frac{B}{(x-a)^p},\quad \frac{Cx+D}{x^2+mx+n},\quad \frac{Ex+F}{(x^2+mx+n)^q},$$

where A, B, C, D, E, F are constants, p, q positive integers, and x^2+mx+n an expression without real linear factors.* With this assumption we shall show how to decompose certain classes of fractions.

* See Chrystal's *Algebra*, Fifth edition, Part I, Chapter VIII.

125. Case I. *When the denominator can be resolved into factors of the first degree, all of which are real and different.*

Example: Resolve $\dfrac{1-x+6x^2}{x-x^3}$ into its simplest partial fractions.

The sum of three fractions

$$\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$$

will give a fraction whose denominator is $x-x^3$. We, therefore, try to determine A, B, and C so that

$$\frac{1-x+6x^2}{x-x^3}\equiv\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$$
$$\equiv\frac{A(1-x)(1+x)+Bx(1+x)+Cx(1-x)}{x(1+x)(1-x)}.$$

Then, $1-x+6x^2\equiv A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$. (1)

The two members of (1) are equal for all values of x except possibly for $x=0$, $x=1$, $x=-1$. Hence, by Art. 94, Corollary II, they are equal for these values. In (1), making

$x=0$, we obtain $A=1$;

making $x=1$, we obtain $B=3$;

making $x=-1$, we obtain $C=-4$.

Therefore, $$\frac{1-x+6x^2}{x-x^3}\equiv\frac{1}{x}+\frac{3}{1-x}-\frac{4}{1+x}.$$

The values of A, B, and C could also have been obtained by arranging the right-hand member of (1) in powers of x and equating coefficients of like powers (Art. 94, Corollary I); thus,

$$1-x+6x^2=A+(B+C)x+(-A+B-C)x^2.$$
$$A=1,$$
$$B+C=-1,$$
$$-A+B-C=6.$$

These equations when solved yield $A=1$, $B=3$, $C=-4$.

In resolving a fraction into partial fractions, for every factor $(x-a)$ occurring in the denominator there is a single partial fraction of the form $\dfrac{A}{x-a}$ where A is a constant.

Exercise. Resolve $\dfrac{5x+1}{x^2-2x-35}$ into partial fractions.

126. CASE II. *When the denominator can be resolved into real linear factors, some of which are repeated.*

EXAMPLE: Resolve $\dfrac{6x^3-8x^2-4x+1}{x^2(x-1)^2}$ into its simplest partial fractions.

The sum of four fractions

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$$

will give a fraction whose denominator is $x^2(x-1)^2$; we therefore try to determine A, B, C, D so that

$$\frac{6x^3-8x^2-4x+1}{x^2(x-1)^2}\equiv\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}.$$

Then,

$$\begin{aligned}6x^3-8x^2-4x+1 &\equiv Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2\\ &\equiv(A+C)x^3+(-2A+B-C+D)x^2\\ &\qquad+(A-2B)x+B.\end{aligned}$$

Equating coefficients of like powers (Art. 94, Corollary I) we have

$$\begin{aligned}A+C&=6,\\ -2A+B-C+D&=-8,\\ A-2B&=-4,\\ B&=1.\end{aligned}$$

Solving these equations for A, B, C, D, we find

$$A=-2,\ B=1,\ C=8,\ D=-5.$$

Hence, $\dfrac{6x^3-8x^2-4x+1}{x^2(x-1)^2}\equiv\dfrac{-2}{x}+\dfrac{1}{x^2}+\dfrac{8}{x-1}-\dfrac{5}{(x-1)^2}.$

In this case, for every factor $(x-a)$ which occurs r times there are r partial fractions of the form

$$\frac{A_1}{x-a},\ \frac{A_2}{(x-a)^2},\ \cdots,\ \frac{A_r}{(x-a)^r},$$

where $A_1, A_2, \cdots, A_r$ are constants.

Exercise. Separate $\dfrac{3x^3+4x^2+8x+2}{x(x+3)^3}$ into partial fractions.

127. CASE III. *When the denominator contains quadratic factors which are not repeated and which cannot be separated into real linear factors.*

EXAMPLE: Resolve $\dfrac{3x^2-2}{(x^2+x+1)(x+1)}$ into a sum of partial fractions.

Let $$\frac{3x^2-2}{(x^2+x+1)(x+1)} \equiv \frac{Ax+B}{x^2+x+1} + \frac{C}{x+1}.$$

Then, $$3x^2-2 \equiv (Ax+B)(x+1)+C(x^2+x+1),$$
$$\equiv (A+C)x^2+(A+B+C)x+B+C.$$

Equating coefficients of like powers, we have

$$A+C=3,$$
$$A+B+C=0,$$
$$B+C=-2,$$

whence, $$A=2,\ B=-3,\ C=1;$$

and $$\frac{3x^2-2}{(x^2+x+1)(x+1)} \equiv \frac{2x-3}{x^2+x+1} + \frac{1}{x+1}.$$

In this case, for every factor x^2+mx+n occurring once, there is a single partial fraction of the form $\dfrac{Ax+B}{x^2+mx+n}$, where A and B are constants.

Exercise. Resolve $\dfrac{3x^3-4x^2+6x-1}{(x^2+x+1)(x^2-x+1)}$ into partial fractions.

128. CASE IV. *When the denominator contains quadratic factors which are repeated.*

EXAMPLE: Resolve $\dfrac{x^4+x^3-2x^2-5x-4}{(x-1)(x^2+x+1)^2}$ into partial fractions.

Let $$\frac{x^4+x^3-2x^2-5x-4}{(x-1)(x^2+x+1)^2} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}.$$

then,

$$x^4+x^3-2x^2-5x-4 \equiv A(x^2+x+1)^2+(Bx+C)(x-1)(x^2+x+1)$$
$$+(Dx+E)(x-1)$$
$$\equiv (A+B)x^4+(2A+C)x^3+(3A+D)x^2$$
$$+(2A-B-D+E)x+(A-C-E).$$

Equating coefficients, we have:

$$\begin{aligned} A + B &= 1, \\ 2A + C &= 1, \\ 3A + D &= -2, \\ 2A - B - D + E &= -5, \\ A - C - E &= -4. \end{aligned}$$

Solving these equations for A, B, C, D, E, we find $A = -1$, $B = 2$, $C = 3$, $D = 1$, $E = 0$. Hence,

$$\frac{x^4 + x^3 - 2x^2 - 5x - 4}{(x-1)(x^2+x+1)^2} \equiv -\frac{1}{x-1} + \frac{2x+3}{x^2+x+1} + \frac{x}{(x^2+x+1)^2}.$$

In this case, for every factor $(x^2 + mx + n)$ occurring r times, there are r partial fractions of the form,

$$\frac{A_1 x + B_1}{x^2 + mx + n}, \quad \frac{A_2 x + B_2}{(x^2 + mx + n)^2}, \quad \cdots, \quad \frac{A_r x + B_r}{(x^2 + mx + n)^r},$$

where A_1, A_2, $\cdots$, A_r, B_1, B_2, $\cdots$, B_r are constants.

Exercise. Resolve $\dfrac{2x^4 - 5x^3 + 15x^2 + 21x + 16}{x(x^2 - 2x + 4)^2}$ into partial fractions.

EXERCISES

Resolve into the simplest partial fractions:

1. $\dfrac{5}{x^2 - x - 6}$.

2. $\dfrac{x+4}{2x - x^2 - x^3}$.

3. $\dfrac{x}{x^2 - 2x - 3}$.

4. $\dfrac{5x^2 - 6x - 5}{(x-1)^3(x+2)}$.

5. $\dfrac{2x - 5}{(x-1)(x-2)}$.

6. $\dfrac{x^2 + 1}{x(x-1)(x-2)}$.

7. $\dfrac{5x+1}{x^2 - 1}$.

8. $\dfrac{x^2 + 1}{x(x-1)^2}$.

9. $\dfrac{x^2 - 4x + 5}{(x-1)^2(x^2+1)}$.

10. $\dfrac{x^4 + x^3 + 2x^2 - x - 1}{x^3 - x}$.

11. $\dfrac{x+4}{(x-1)(x^2 - 5x + 6)}$.

12. $\dfrac{5x^2 + 8x + 11}{(x^2+1)(x+1)(x-3)}$.

13. $\dfrac{2x^2 - 5x + 7}{(x-3)(x^2 - 3x + 2)}$.

14. $\dfrac{6(2x-1)}{x(x+2)(x-3)}$.

15. $\dfrac{x^2 + x - 3}{(x^2 - 4x + 3)(x-2)}$.

16. $\dfrac{4}{x^3 + 4x}$.

17. $\dfrac{4}{x^3 - 1}$.

18. $\dfrac{x^2 + 1}{x^3(x-1)}$.

19. $\dfrac{1 + 7x - x^2}{(1+3x)^2(1-10x)}$.

20. $\dfrac{2x+5}{(x-1)^3(x-3)}$.

21. $\dfrac{5-9x}{(1-3x)^3(1+x)}$.

22. $\dfrac{9-2x}{(x+2)(x^2-2x+5)}$.

23. $\dfrac{x^4-2x^3+3x^2-x}{(x-1)(x^2-x+1)^2}$.

24. $\dfrac{6x^3+2x^2+2x-2}{x^4-1}$.

25. $\dfrac{2x^3-8x^2-7x+1}{x^4+x^3-x-1}$.

26. $\dfrac{4x^3-18x^2}{(2x-3)^4}$.

27. $\dfrac{3x-1}{x^2(x+1)^2}$.

28. $\dfrac{20x^2-2x^3}{x^4-16}$.

29. $\dfrac{x^3-2x^2-6x-21}{x^2-4x-5}$.

30. $\dfrac{5x^2-4x+16}{(x-3)(x^2-x+1)^2}$.

31. $\dfrac{6x^3-8x^2-4x+1}{x^2(x-1)^2}$.

32. $\dfrac{3x^3+19x^2+35x}{(x+2)^3}$.

33. $\dfrac{45+36x-x^2}{x^4-6x^2-27}$.

34. $\dfrac{x^2+x+2}{(x-1)^2(x^2-x+1)}$.

35. $\dfrac{x^3-2}{(x^2+x+1)(x^2+x+2)^2}$.

36. $\dfrac{x^3+2x^2+2}{(x^2+1)^2}$.

37. $\dfrac{17-11x+7x^2}{(1+x+x^2)(1-x)^3}$.

38. $\dfrac{4x^4+8x^3+6x^2+6x+5}{(3x+2)(x^2+1)^2}$.

CHAPTER XVI

PERMUTATIONS AND COMBINATIONS

129. Introduction. Two positions are to be filled in an office — one that of stenographer and the other that of messenger. There are 12 applicants for the position of stenographer, and 3 for that of messenger. In how many ways can the two positions together be filled?

The position of stenographer can be filled in 12 ways, and with each of these there is a choice of 3 messengers. Hence, the two positions can be filled in $12 \times 3 = 36$ ways.

This example illustrates the following

FUNDAMENTAL PRINCIPLE. *If one thing can be done in m different ways; and if, after this is done in one of these ways, a second thing can be done in n ways, then the two together can be done in the order stated in mn ways.*

For, corresponding to each of m ways of doing the first thing, there are n ways of doing the second thing. In other words, there are n ways of doing the two together for each way of doing the first thing. Hence, there are in all mn ways of doing the two things together.

A convenient and evident extension of the fundamental principle may be stated in the following form:

If one thing can be done in m_1 ways, a second in m_2 ways, a third in m_3 ways, and so on, the number of different ways in which they can be done when taken all together in the order stated is $m_1 m_2 m_3 \cdots$.

130. Meaning of a permutation. *Each different arrangement which can be made of all or part of a number of things is called a* **permutation.**

By the expression "number of permutations of n things taken r at a time" is meant the number of permutations consisting of r

things each which can be formed from n different things. Thus, the permutations of the letters abc taken all at a time are

$$a\,b\,c,\quad a\,c\,b,\quad b\,a\,c,\quad b\,c\,a,\quad c\,a\,b,\quad c\,b\,a$$

The permutations of the four letters $a\,b\,c\,d$ taken three at a time are;

$a\,b\,c$	$b\,a\,c$	$c\,a\,b$	$d\,a\,b$
$a\,c\,b$	$b\,c\,a$	$c\,b\,a$	$d\,b\,a$
$a\,c\,d$	$b\,c\,d$	$c\,b\,d$	$d\,b\,c$
$a\,d\,c$	$b\,d\,c$	$c\,d\,b$	$d\,c\,b$
$a\,b\,d$	$b\,a\,d$	$c\,a\,d$	$d\,a\,c$
$a\,d\,b$	$b\,d\,a$	$c\,d\,a$	$d\,c\,a$

131. Permutations of things all different. The special cases just considered lead us to the problem of deriving a formula for the number of permutations of n things taken r at a time. The symbol ${}_nP_r$ is used to represent this number.

The number of permutations of n different things taken r at a time is

$$_nP_r = n(n-1)\cdots(n-r+1).$$

The number ${}_nP_r$ required is the same as the number of ways of filling r different positions with n different things. We may represent the n things by $a_1, a_2, \cdots, a_n$ and ask how many permutations of r letters can be formed from them. For the first place there is a choice of n letters, for the second a choice of $n-1$, for the third of $n-2$, and so on. For the rth place there is then a choice of $n-r+1$ letters. It follows (Art. 129) that

$$_nP_r = n(n-1)\cdots(n-r+1). \tag{1}$$

When $r = n$, (1) becomes

$$_nP_n = n(n-1)\cdots 2\cdot 1 = n!. \tag{2}$$

That is, the number of permutations of n things taken n at a time is n!

132. Permutations of n things not all different. Consider the number of permutations of the letters in the word *book*. It gives no new permutation to interchange the o's. Let P be the

number of permutations. If we should replace oo by dissimilar characters o_1o_2, there would be $2!$ permutations of o_1o_2 corresponding to each of the P permutations. But if the letters were all different the number of permutations would be $4!$. Hence,

$$4! = 2!\ P, \quad P = \frac{4!}{2!} = 12.$$

This example illustrates the

THEOREM. *If P is the number of permutations of n things taken all at a time, of which n_1 are alike, n_2 others alike, n_3 others alike, and so on, then*

$$P = \frac{n!}{n_1!\, n_2!\, n_3! \cdots}.$$

To establish the theorem, suppose we should replace n_1 like things by n_1 unlike things, there would be $P \cdot n_1!$ permutations obtained from the original P permutations. In each of these permutations there would be n_2 things alike, and n_3 others alike. Similarly, replacing the n_2 like things by n_2 dissimilar things, we get $P \cdot n_1! \cdot n_2!$ permutations in each of which there would be n_3 alike. Continuing this argument, we find that the number of permutations of n things taken all at a time, when n_1 are alike, n_2 others alike, n_3 others alike, and so on, is given by

$$P = \frac{n!}{n_1!\, n_2!\, n_3!}.$$

EXERCISES AND PROBLEMS

1. How often can 6 ball players take seats together on a bench without sitting twice in the same order?

2. In how many different orders can the colors violet, indigo, blue, green, yellow, orange, and red be arranged when taken all together?

3. How many different permutations can be made of the letters of the word "stone" when taken 3 at a time?

4. Five different positions are to be filled, and there are 20 applicants each applying for any one of the positions. In how many ways can the positions be filled?

5. In how many ways can ten books be arranged on a shelf if the places of two of them are fixed?

6. Given ${}_nP_4 = 5\,{}_nP_3$, find n.

7. How many permutations can be made of the letters of the word "Illinois"? Of the word "Mississippi"?

8. In how many ways may a party of 8 people take their places at a round table?

9. How many different combinations may be struck from 8 bells if only 3 are struck at one time?

10. Four persons enter a railway carriage in which there are 6 seats. In how many ways can they take their places?

11. Find the number of permutations of letters in the word "level."

12. How many different numbers of six figures each can be formed by permuting the figures 233455?

13. Write all the permutations of the letters *abcd*, when taken (1) two at a time, (2) four at a time.

133. Combinations. *A set of things or elements without reference to the order of individuals within the set is called a* **combination.**

Thus, *abc, acb, bac, bca, cab, cba* are the same combination. By the "number of combinations of n things taken r at a time" is meant the number of combinations of r individuals which can be formed from n things.

Thus, the combinations of a b c taken two at a time are *ab, ac, bc.*

134. Combinations of things all different. Let ${}_nC_r$ denote the number of combinations of n things taken r at a time. Then a formula can be derived for ${}_nC_r$ by establishing the relation between ${}_nC_r$ and ${}_nP_r$.

Take one combination of r things; with this $r!$ permutations can be made. Take a second combination; with this $r!$ permutations can be made. There are thus $r!$ permutations for each combination. Hence, there are in all ${}_nC_r\, r!$ permutations of n things taken r at a time. That is,

$$ {}_nC_r \cdot r! = {}_nP_r,$$

whence

$$ {}_nC_r = \frac{{}_nP_r}{r!}.$$

Since $${}_nP_r = n(n-1)\cdots(n-r+1), \qquad \text{(Art. 131)}$$

we have $$\boldsymbol{{}_nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!}}.$$

Multiplying numerator and denominator by $(n-r)!$, we get

$$\boldsymbol{{}_nC_r = \frac{n!}{r!\,(n-r)!}}.$$

Since $${}_nC_{n-r} = \frac{n(n-1)\cdots(r+1)}{(n-r)!} = \frac{n!}{(n-r)!\,r!},$$

it follows that the number of combinations of n things taken r at a time is the same as the number taken $n-r$ at a time.

135. Binomial coefficients. It may be noted that the formula for ${}_nC_r$ is the coefficient of the $(r+1)$st term of the binomial expansion $(a+x)^n$. The binomial theorem for positive integral exponents may therefore be written in the form

$$(a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots + {}_nC_{n-1}ax^{n-1} + {}_nC_nx^n.$$

136. Total number of combinations. *The total number of combinations of n things taken 1, 2, 3, $\cdots$, n at a time is $2^n - 1$.* If we write the binomial theorem as in the last section, we obtain

$$(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_{n-1}x^{n-1} + {}_nC_nx^n.$$

Putting $x = 1$, we get

$$\boldsymbol{2^n - 1 = {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-1} + {}_nC_n}.$$

EXERCISES AND PROBLEMS

1. A woman with 10 friends to invite can have how many dinner parties with 6 guests without having the same company of 6 twice?

2. A man and his wife wish to invite 4 men and 5 women to dinner, but find they can entertain only 6 guests at one dinner. In how many ways can they invite 3 men and 3 women out of this group?

3. A man has 5 friends. In how many ways can he invite one or more of them to dinner?

4. How many different assemblages of 1000 persons can be selected from an assemblage of 1002 persons?

5. In a certain town, there are 4 aldermen to be elected, and there are 8 candidates. How many different tickets can be made up?

6. Find ${}_{20}C_{17}$; ${}_{12}C_{9}$.

7. Given ${}_{n}C_{4} = 210$, find n.

8. Given ${}_{n}P_{r} = 272$, and ${}_{n}C_{r} = 136$, find n and r.

9. How many different sums of money can be formed with a penny, a nickel, a dime, a quarter, and a half dollar ?

10. How many different sums may be formed with a penny, a nickel, a dime, a quarter, a half dollar, and a dollar ?

11. There are five letter boxes in a town. In how many ways can a person post two letters ?

12. In how many ways can 5 books be selected from a set of 11 ?

13. A committee of 6 is to be chosen from 7 Englishmen and 4 Americans. If the committee is to contain at least 2 Americans, in how many ways may the committee be chosen ?

14. Prove that ${}_{n}C_{r} = {}_{n}C_{n-r}$.

15. Make use of the theorem ${}_{n}C_{r} = {}_{n}C_{n-r}$ to evaluate ${}_{100}C_{98}$.

16. Out of 15 consonants and 4 vowels how many words can be formed each containing 3 consonants and 2 vowels ?

17. How many straight lines can be drawn through pairs of points selected from 10 points no three of which are in the same straight line ?

18. In how many ways can a pack of 52 playing cards be divided into 4 hands, the order of the hands, but not the cards in the hands, to be regarded?

19. There are 5 trails to the top of a mountain. In how many ways may a person go up and return by a different trail ?

20. In how many ways can 8 books be arranged on a shelf so that two particular books will not be together ?

21. How many baseball teams of 9 men each can be chosen from 15 players of whom 8 are qualified to play in the infield only, 5 in the outfield only, and 2 in any position (battery included in infield) ?

22. How many different combinations can be formed with the following weights ?

1 decigram,	1 gram,	1 10-gram,
1 2-decigram,	1 2-gram,	1 20-gram,
1 3-decigram,	1 3-gram,	1 30-gram,
1 5-decigram,	1 5-gram,	1 50-gram.

23. On how many nights may a different guard of 4 men be posted out of 16 soldiers ? On how many of these nights will any particular man be on guard ?

24. With 4 white balls, 6 black balls, and 9 red balls, how many different combinations can be formed each consisting of 1 white ball, 3 black balls, and 5 red balls?

25. A town which has 11 physicians, 13 teachers, and 8 lawyers can form how many committees each consisting of 3 physicians, 4 teachers, and 2 lawyers?

26. A company consists of 100 soldiers. In how many ways is it possible to leave 60 men to garrison a fort and to divide the remainder into two scouting parties of 20 men each?

CHAPTER XVII

PROBABILITY

137. Meaning of probability. If a bag contains three white and five black balls and one ball is drawn out at random, what is the probability that this ball is white?

The **event** in question is said to happen if a white ball is drawn, and to fail if a black ball is drawn. The number of ways in which the event may happen is 3, and the total number of possible ways in which it may happen and fail is 8. For this reason, $\frac{3}{8}$ is said to be the probability of drawing a white ball. This illustrates the following definition of probability:

If all the happenings and failings of an event can be analyzed into $r+s$ possible ways each of which is equally likely; and if in r of these ways the event will happen, and in s of them fail, the probability that the event will happen is $\frac{r}{r+s}$ and the probability that it will fail is $\frac{s}{r+s}$.

COROLLARY. *The sum of the probability that an event will happen and the probability that it will fail is* 1, *which is the symbol for certainty.*

In applying the definition of probability, the fact should not be overlooked that the ways are assumed to be "equally likely." To illustrate the need of precaution in this matter, consider the following

EXAMPLE: What is the probability that a man, A, in good health will die within the next 24 hours?

We might argue that the event can happen in only one way and fail in only one way, and that the probability that A will die in the next 24 hours is therefore $\frac{1}{2}$. What is the flaw in this argument?

The expression "equally likely" indicates that we have no more reason to expect the event to take place in one way than in any other.

138. Probability derived from observation. *If it be observed that an event has happened m times in n possible cases (n a large number); then, in the absence of further knowledge, it is assumed that the best estimate of the probability that the event will happen on a given occasion in question is $\frac{m}{n}$, and that confidence in this estimate increases as n increases.*

Such estimates of probability are of much practical value in insurance and statistics. For example, according to the American Experience Table of Mortality, of 85,441 men living at the age of 30, the number living ten years later is 78,106. The probability that a man aged 30 will live ten years is taken to be $\frac{78106}{85441}$.

139. Expectation of money. *If p is the probability that a person will win a sum of money m, we may define his expectation as pm.*

PROBLEMS

1. A bag contains ten times as many white balls as black balls, and one ball is to be drawn out at random. What is the probability that the ball drawn is white?

2. Five coins are tossed. What is the probability that exactly two of them are heads?

SOLUTION: Since each coin can fall in two ways, the five can fall in $2^5 = 32$ ways. The two coins can be selected from the five in ${}_5C_2 = 10$ ways. Hence, the probability is $\frac{10}{32}$.

3. From a bag containing 5 black and 4 white balls, 3 are drawn at random. Find the probability that 2 are black and 1 is white.

4. If from a suit of 13 cards, 2 cards are drawn, what is the probability that an ace and a king are drawn?

5. A gambler is to win \$ 30 if an ace is thrown with a single die; what is the value of his expectation?

6. According to a mortality table, it appears that of 100,000 persons at the age of ten years, only 69,804 reach the age of fifty. Find the probability that a child aged ten will reach the age of fifty years.

7. From a committee of 3 sophomores, 4 juniors, and 5 seniors, a subcommittee of 4 is selected by lot. Find the probability that it will consist: (1) of 2 juniors and 2 seniors; (2) of 1 sophomore, 1 junior, and 2 seniors; (3) of 4 seniors.

8. From a bag containing 10 five-dollar bills and 20 two-dollar bills, I have the privilege of drawing a bill at random. What is the value of my expectation?

140. Events of a set are said to be **independent** or **dependent** according as the occurrence of any one of them does not or does affect the occurrence of others in the set. They are said to be **mutually exclusive** when the occurrence of any one of them on a particular occasion excludes the occurrence of any other on that occasion.

Independent events. *The probability that all of a set of independent events will happen on a given occasion when all of them are in question is the product of their separate probabilities.*

Let P be the probability that all events of the set will happen and $p_1, p_2, \cdots p_r$ be their separate probabilities. It is to be proved that $P = p_1 p_2 \cdots p_r$. Suppose the event corresponding to p_1 can happen in a_1 ways and fail in b_1 ways: the event corresponding to p_2 can happen in a_2 ways and fail in b_2 ways; and so on.

Then, $$p_1 = \frac{a_1}{a_1 + b_1},\ p_2 = \frac{a_2}{a_2 + b_2},\ \cdots,\ p_r = \frac{a_r}{a_r + b_r}.$$

By the fundamental principle (Art. 129) all the separate events can happen together in $a_1 a_2 \cdots a_r$ ways out of

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_r + b_r)$$

possible ways of happening and failing.

Hence, $$P = \frac{a_1 a_2 \cdots a_r}{(a_1 + b_1)(a_2 + b_2) \cdots (a_r + b_r)} = p_1 p_2 \cdots p_r.$$

Dependent events. *If the probability of a first event is p_1, and if, after this has happened, the probability of a second event is p_2; then the probability that both events will happen in the order specified is $p_1 p_2$.* The extension to any number of events is obvious.

Exclusive events. *If the separate probabilities of r mutually exclusive events be $p_1, p_2, \cdots, p_r$, the probability that one of these events will happen on a particular occasion when all of them are in question is $p_1 + p_2 + \cdots + p_r$.*

This proposition may be regarded as an immediate consequence of the definition of probability for mutually exclusive events. To illustrate, the probability of throwing an ace or a deuce in single throw is clearly $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

PROBLEMS

1. If the probability is $\frac{1}{4}$ that the age of a man selected at random from a group of men is between 20 and 25 years, and $\frac{1}{3}$ that it is between 25 and 35, what is the probability that his age is between 20 and 35 years?

2. What is the probability of throwing an ace with a single die in two trials?

3. The probability that A will live ten years is $\frac{1}{5}$ and the probability that B will live ten years is $\frac{1}{4}$. What is the probability that they will both live ten years?

4. Find the probability of drawing 2 white balls in succession from a bag containing 5 white and 6 black balls if the first ball drawn is not replaced before the second drawing is made.

5. One purse contains 9 coins consisting of 2 dimes, 3 quarters, and 4 half dollars. If one coin is drawn at random from the purse, what is the probability of its being either a quarter or a half dollar?

6. In a bag are 4 white and 6 black balls; find the chance that out of 5 drawn, 2 and only 2 are white.

7. Find the probability of throwing at least 8 in a single throw with two dice.

8. A traveler has three railroad connections to make. If the probability is $\frac{2}{3}$ that he would make any particular connection taken alone, what is the probability of his making all three connections?

9. The probability that a man of a certain age will die within 20 years is 0.2, and that his wife will die within that time is 0.15. What is the probability that at the end of 20 years (1) both will be dead? (2) both will be living? (3) the man will be living and his wife dead? (4) the man will be dead and his wife living?

141. Repeated trials. *If p is the probability that an event will happen in any single trial, then ${}_nC_r p^r q^{n-r}$ is the probability that this event will happen exactly r times in n trials, where $q = 1 - p$ is the probability that the event will fail in any single trial.*

For, the probability that it will happen in r specified trials and fail in the remaining $n - r$ is $p^r q^{n-r}$ (Art. 140), and r trials can be selected from n trials in ${}_nC_r$ ways. These ways being mutually exclusive, we have, by Art. 140, that the probability in question is

$$ {}_nC_r p^r q^{n-r}. $$

It will be observed that ${}_nC_r p^r q^{n-r}$ is the $(n - r + 1)$th term of the binomial expansion of $(p + q)^n$.

We next inquire into the probability that an event such as is described above happens at least r times in n trials. The event happens *at least* r times if it happens exactly n, $n - 1$, $n - 2$, $\cdots$, or r times in n trials.

Hence, we have the following

THEOREM: *The probability that an event will happen at least r times in n trials is* $p^n + {}_nC_{n-1} p^{n-1} q + {}_nC_{n-2} p^{n-2} q^2 + \cdots + {}_nC_{n-r} p^r q^{n-r}$.

This expression is the first $n - r + 1$ terms of the binomial expansion of $(p + q)^n$.

PROBLEMS

1. In tossing a coin, what is the probability that in six tosses (1) exactly three result in heads; (2) at least three result in heads?

2. According to the American Experience Table of Mortality, out of 100,000 persons living at the age of 10 years, 91,914 are living at the age of 21 years. Each of five boys is now 10 years old; what is the probability that exactly four of them will live to be 21? That at least four of them will live to be 21?

3. A's chance (probability) of winning any single game against B is $\frac{3}{4}$. Find the chance of his winning at least three games out of seven.

4. In tossing ten coins, what is the probability that at least four of them will be heads?

5. Find the expectation of a man who buys a lottery ticket in a lottery of 100 tickets where there are four prizes of \$100, ten of \$50, and twenty of \$5.

6. If, in the long run, one vessel out of every 50 is wrecked, find the probability that of 6 vessels expected (1) exactly 5 will arrive safely, (2) at least 5 will arrive safely.

7. Which is the greater, the probability of throwing at least one ace in six trials of throwing a die, or the probability of throwing at least one head of a coin in two trials?

8. According to the American Experience Table of Mortality, out of 89,032 persons living at the age of 25 years, 26,237 will be living at the age of 75. A husband and wife are 25 each at the date of marriage; what is the probability that they will live to celebrate their golden wedding? What is the probability that at least one of them will be living 50 years after the marriage?

9. A machinist works 300 days in a year. If the probability of his meeting with an accident on any particular work day is $\frac{1}{1000}$, show that the probability of his entirely escaping injury for a year is approximately $\frac{3}{4}$. (Use logarithms.)

10. A card is to be drawn from a whist deck and replaced by a joker, and then a second card is to be drawn. What is the probability that both cards drawn will be aces?

11. A bag contains 6 balls. A part or all of the balls are drawn. Under the condition that the ways in which r balls can be drawn from 6 balls are equally likely for $r = 1, 2, 3, \cdots, 6$, what is the probability of drawing an even number of balls?

12. An Italian nobleman, interested in gambling, had, by continued observation of a game with three dice, noticed that the sum 10 appeared more often than the sum 9. He expressed his surprise at this to Galileo and asked for an explanation. Find the probability of (1) the sum 10, (2) the sum 9, and explain the difficulty of the nobleman.

13. A and B take turns in throwing with a single die, A throwing first. The one who throws an ace first is to receive a prize of \$66. What are the values of their expectations?

14. In the population of continental United States as given in the census of 1900 there were 75,994,575 persons of whom 64,763 were blind and 89,287 were deaf. What is the best estimate from these figures of the probability that a person chosen at random from such a population would be (1) blind? (2) deaf? (3) both blind and deaf if the two were independent? Estimate to the nearest integer the number in the total population that would be both blind and deaf if it were correct to assume the two defects independent.

CHAPTER XVIII

DETERMINANTS

142. Extension of the determinant notation. Determinants of the second and third orders were used in Chapter V in the solution of systems of linear equations in two and three unknowns; and a determinant of the second order was so defined that the pair of values

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{1}$$

satisfies the system of equations,

$$a_1 x + b_1 y = c_1,$$
$$a_2 x + b_2 y = c_2,$$

provided

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \tag{2}$$

Analogously, a determinant of the third order was so defined that the set of values

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \tag{3}$$

satisfies the system of equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

provided

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0. \tag{4}$$

The determinant notation is extended in the present chapter to the solution of systems of linear equations containing more than three unknowns, and to certain problems of elimination.

It will be observed that each term (*e.g.* a_1b_2 and $a_1b_2c_3$) in the expansions,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \tag{5}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2, \tag{6}$$

of determinants of orders 2 and 3 respectively, consists (except for sign) of the product formed by taking one and only one element from each row and column. This fact suggests the extension of determinants to represent certain expressions in n^2 elements by means of an array,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \cdots & l_1 \\ a_2 & b_2 & c_2 & d_2 \cdots & l_2 \\ a_3 & b_3 & c_3 & d_3 \cdots & l_3 \\ a_4 & b_4 & c_4 & d_4 \cdots & l_4 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot & \cdot \\ a_n & b_n & c_n & d_n \cdots & l_n \end{vmatrix} \tag{7}$$

where the expansion is to consist of terms which are products formed by taking one and only one element from each row and column, and where the signs of terms are to be consistent with the special cases of $n = 2$ and $n = 3$.

A determinant such as (7) which has n rows and n columns is called a determinant of the nth order. The diagonal $a_1b_2c_3 \cdots l_n$ is called the principal diagonal.

To fix the signs of terms in the expansion of a determinant of any order, the notion of an inversion is introduced. If, in an arrangement of positive integers, a greater precedes a less, there is said to be an **inversion.** Thus, in the order 12543, there are three inversions: 5 before 4, 5 before 3, 4 before 3. In 2341576, there are four inversions. When applied to any term in the expansion of a determinant such as (7), we say there is an inversion if the order of the subscripts presents an inversion when the letters (apart from subscripts) have the order $abcd\cdots l$ of the principal diagonal. With respect to determinants of orders 2 and 3, it may be observed that the number of inversions is even when the term is positive, and that the number of inversions is odd when the term is negative.

Consistently with these conditions, we lay down the following

Definition. *A square array of n^2 elements, such as has been considered in the cases $n=2$ and $n=3$, is called* **a determinant of the nth order.** *It is an abbreviation for the algebraic sum of all the different products that can be formed by taking as factors one and only one element from each column and each row of the array, and giving to each term a positive or a negative sign according as the number of inversions of the subscripts of the term is even or odd, when the letters have the same order as in the principal diagonal.*

It may be added that if in any case the number of inversions in the principal diagonal is different from zero, the sign of a term is + or − according as the number of inversions in its subscripts differs from the number in the principal diagonal by an even or odd number. Since the subscripts fix the signs of terms, it may appear necessary to carry subscripts along in any numerical case, but we shall derive other modes of expansion (Art. 144) which make this unnecessary. We shall, in general, use the Greek letter Δ to represent a determinant.

143. Properties of determinants. The following theorems embody the most important properties of determinants.

I. *The expansion of a determinant Δ of order n contains $n!$ terms.*

Since the number of terms is the same as the number of permutations of the subscripts 1, 2, 3, $\cdots$, n, the number is $n!$ (Art. 131).

II. *If in a determinant* Δ *corresponding rows and columns are interchanged, the expansion is unchanged.*

Thus,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

III. *If two rows (or columns) of a determinant* Δ *are interchanged, the sign of the determinant is changed.*

Let us take for simplicity a determinant of the third order, but the argument used will clearly apply to any determinant. Thus,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}.$$

In the first place, interchanging two adjacent rows will simply interchange two adjacent subscripts in each term of the expansion. This will change the sign of every term of the expansion. Consider next the effect of interchanging any two rows (or columns) separated by m intermediate rows. The lower row can be brought just below the upper one by m interchanges of adjacent rows. To bring likewise the upper row into the original position of the lower row, $m+1$ further interchanges are necessary. Hence, interchanging the two rows in question is equivalent to $2m+1$ interchanges of adjacent rows. Since $2m+1$ is an odd number, this process changes the sign of the determinant.

IV. *If a determinant* Δ *has two rows (or columns) identical, its value is zero.*

If we interchange two rows, we obtain, by III, $-\Delta$. But since the interchange of two identical rows does not alter the determinant we have
$$\Delta = -\Delta,$$
that is,
$$2\Delta = 0,$$
or
$$\Delta = 0.$$

V. *If all the elements of a row (or column) of* Δ *are multiplied by the same number* m *the determinant is multiplied by* m.

For, one element from the column multiplied by m must enter into each term of the expansion of Δ.

VI. *If one row (or column) of* Δ *has as elements the sum of two or more numbers,* Δ *can be written as the sum of two or more determinants.* That is,

$$\Delta = \begin{vmatrix} a_1+a_1'+a_1'' & b_1 & c_1 \\ a_2+a_2'+a_2'' & b_2 & c_2 \\ a_3+a_3'+a_3'' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.$$

This theorem is evident for this special case, since each term in the expansion of Δ is evidently equal to the sum of the corresponding terms of the three determinants. Similarly, we can prove the general case.

VII. *The value of any determinant* Δ *is not changed if each element of any row (or column), or each element multiplied by any given number, m, be added to the corresponding element of any other row (or column).*

By V and VI,

$$\begin{vmatrix} a_1+ma_3 & a_2 & a_3 \\ b_1+mb_3 & b_2 & b_3 \\ c_1+mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + m^* \begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix},$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0, \text{ by IV.}$$

Likewise, the theorem can be proved for a determinant of any order.

144. Development by minors. If we suppress both the row and column to which any element, say c_k, of the determinant belongs, the unsuppressed elements form a determinant called the first minor of c_k, and which we shall denote by the capital letter C_k. Thus, in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

* This notation means that the determinant is multiplied by m.

A determinant Δ may be expressed in terms of the elements $c_1, c_2, \cdots, c_n$ of a column (or row) and their first minors as follows:

Form the product of each element such as c_k in the column by the corresponding minor C_k. Give each of the products thus formed a positive or a negative sign according as the sum of the number of the row and the number of the column containing c_k is even or odd, and take the algebraic sum of these results. This sum is equal to Δ.

Thus,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

If we can establish this theorem, we have a systematic method for expanding any determinant, since the first minors of Δ are again determinants which can be expressed in terms of their own minors. This process can be continued until we have the expansion of Δ.

The proof of the theorem involves two steps:

(1) The coefficient of the leading element a_1 in the expansion of Δ is the minor, A_1, of a_1. For, A_1 is a determinant of order $n-1$ in elements $b_2, b_3, \cdots b_n, \cdots$ and its expansion therefore contains a term for each permutation of $2\ 3\ 4 \cdots n$. As to the signs of terms, the number of inversions is not changed by prefixing a_1.

(2) The coefficient of any element c_k in the expansion of Δ is its minor C_k with a + or a − sign, according as the sum of the number of the row and the number of the column containing c_k is even or odd. If c_k is in the hth column and kth row, we can bring it to the leading position (column 1, row 1) without disturbing the relative positions of elements not found in column h or row k. This is done by interchanging the column in which c_k stands with each preceding column in turn until c_k is in column 1, and the row in which c_k stands with each preceding row in turn until c_k is in row 1. In making these changes, the sign of the determinant is, changed, $h-1+k-1=h+k-2$ times (Art. 143, III). Hence, if Δ' denotes this determinant with c_k as the leading letter,
$$\Delta' = (-1)^{h+k-2}\Delta = (-1)^{h+k}\Delta.$$

Let C'_k be the minor of c_k in Δ'. By (1), the sum of the terms in the expansion of Δ' which contain c_k is $c_k C'_k$. Since the minor of c_k in Δ' is the same as in Δ, the coefficient of c_k in the expansion of Δ is $(-1)^{k+k} C_k$. This establishes the second step.

EXERCISES

1. Develop $\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & -4 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{vmatrix}$.

SOLUTION:

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & -4 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 & -4 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -4 \\ -1 & 1 & 1 \end{vmatrix}$$

$$- 1 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -4 \\ 2 & 1 & 0 \end{vmatrix} = 48.$$

2. Develop $\Delta = \begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 4 & 4 & 3 & 2 \\ 5 & 5 & 2 & 1 \end{vmatrix}$.

HINT: Subtracting column 2 from column 1, we have by VII, Art. 143,

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 4 & 3 & 2 \\ 0 & 5 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 3 & 2 \\ 5 & 2 & 1 \end{vmatrix}.$$

3. How many inversions are there in the arrangement 1 4 5 2 3 6 8 7 ?

4. Develop $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & -1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \end{vmatrix}$.

5. Develop $\begin{vmatrix} 2 & 7 & 6 & 5 \\ 1 & 1 & 7 & 3 \\ 1 & 5 & 3 & 4 \\ 4 & 7 & 5 & 6 \end{vmatrix}$.

6. Show that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

HINT: When $a = b$, two columns are identical so that Δ vanishes, and by the factor theorem, Art. 93, $a - b$ is a factor of Δ.

7. Factor $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$ into linear factors.

8. Show that $\Delta = \begin{vmatrix} 1 & a & b \\ 1 & a^2 & b^2 \\ 1 & a^3 & b^3 \end{vmatrix} = ab(a-b)(a-1)(1-b)$.

9. Factor $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$. **10.** Factor $\Delta = \begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}$.

145. We shall now establish a theorem of determinants which is useful in performing the eliminations required in the solution of equations in two or more unknowns.

THEOREM. *In developing a determinant by minors with respect to a certain column (or row), if the elements of this column (or row) are replaced by the corresponding elements of some other column (or row), the resulting expression vanishes.*

For example, we have, by Art. 144,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4.$$

We are to prove that

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 = 0. \qquad (1)$$

The left member of (1) is equal to the expression of the determinant derived from Δ by replacing the column of a's by the b's with corresponding subscripts. But this gives a determinant with two columns identical, which therefore vanishes (Art. 143, IV). The same method of proof can manifestly be applied to a determinant of any order.

146. Systems of linear equations containing the same number of equations as unknowns. In Chapter V, we used determinants to express the solution of simultaneous equations containing two and three unknowns. We are now in a position to make use of determinants to solve a system of n linear equations in n unknowns.

For simplicity of notation, take $n = 4$, and consider the system of equations

$$a_1x + b_1y + c_1z + d_1w = k_1, \tag{1}$$

$$a_2x + b_2y + c_2z + d_2w = k_2, \tag{2}$$

$$a_3x + b_3y + c_3z + d_3w = k_3, \tag{3}$$

$$a_4x + b_4y + c_4z + d_4w = k_4, \tag{4}$$

to be solved for x, y, z, and w if a solution exists. It is convenient to call the determinant of the coefficients of the unknowns,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

the *determinant of the system* of equations.

CASE I. When $\Delta \neq 0$.

As above, let A_1, A_2, $\cdots$, B_1, B_2, $\cdots$ be the minors of a_1, a_2, $\cdots$, b_1, b_2, $\cdots$ respectively. Multiplying both members of (1), (2), (3), and (4) by A_1, $-A_2$, A_3, and $-A_4$, respectively, we obtain

$$A_1a_1x + A_1b_1y + A_1c_1z + A_1d_1w = A_1k_1, \tag{5}$$

$$-A_2a_2x - A_2b_2y - A_2c_2z - A_2d_2w = -A_2k_2, \tag{6}$$

$$A_3a_3x + A_3b_3y + A_3c_3z + A_3d_3w = A_3k_3, \tag{7}$$

$$-A_4a_4x - A_4b_4y - A_4c_4z - A_4d_4w = -A_4k_4, \tag{8}$$

Adding (5), (6), (7), (8), we obtain Δ for the coefficient of x (Art. 144), and zero for coefficients of the other unknowns (Art. 145). That is,

$$\Delta \cdot x = A_1k_1 - A_2k_2 + A_3k_3 - A_4k_4. \tag{9}$$

Similarly,

$$\Delta \cdot y = -B_1k_1 + B_2k_2 - B_3k_3 + B_4k_4, \tag{10}$$

$$\Delta \cdot z = C_1k_1 - C_2k_2 + C_3k_3 - C_4k_4, \tag{11}$$

and

$$\Delta \cdot w = -D_1k_1 + D_2k_2 - D_3k_3 + D_4k_4. \tag{12}$$

If, in Δ, we replace the a's by k's and expand, we have the right-hand member of (9). Similarly, replacing the b's, c's, and d's respectively by k's, we have the right-hand members of (10), (11), and (12). It follows that

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\Delta}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}}{\Delta},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 & d_1 \\ a_2 & b_2 & k_2 & d_2 \\ a_3 & b_3 & k_3 & d_3 \\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}}{\Delta}, \quad w = \frac{\begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}}{\Delta},$$

is a solution, and the only solution, of (1), (2), (3), (4).

The following rule may then be applied to obtain the solution of any system of n linear equations containing n unknowns when Δ, the determinant of the system, is not zero:

Any unknown is equal to a fraction whose denominator is the determinant of the system, and whose numerator is the determinant formed from the determinant of the system by substituting for the coefficients of the unknown sought the corresponding known terms with that sign attached to each known term which it has when on the side of the equation opposite the unknowns.

CASE II. When $\Delta = 0$.

If a solution exists when $\Delta = 0$, it cannot take the preceding form, since division by zero is excluded from algebraic operations. While the theory becomes too complicated in this case to be presented in full here, certain particular cases may well be considered.

As a rule (subject to certain exceptions), a system of equations has no solution when $\Delta = 0$. For example, the system

$$3x + 4y = 5,$$
$$6x + 8y = 9$$

has no solution. Likewise the system

$$x + y - z = 5,$$
$$4x + y - 2z = 9,$$
$$5x + 2y - 3z = 1$$

has no solution.

A system may, however, have an infinite number of solutions when $\Delta = 0$. For instance, the equations

$$x + y - z = 0, \tag{13}$$

$$4x + y - 2z = 0, \tag{14}$$

$$5x + 2y - 3z = 0 \tag{15}$$

constitute such a system. These equations are manifestly satisfied by $x = y = z = 0$. To obtain other solutions solve (13) and (14) for x and y in terms of z. This gives

$$x = \tfrac{1}{3}z, \qquad y = \tfrac{2}{3}z. \tag{16}$$

These values of x and y satisfy (15) as well as (13) and (14). Hence any value assigned to z with the corresponding values x and y obtained from $x = \frac{1}{3}z$, $y = \frac{2}{3}z$ satisfies (13), (14), and (15). Since z may have any value, there is an infinite number of solutions of the system in question.

Systems with an infinite number of solutions may be more generally illustrated by the homogeneous* equations

$$a_1x + b_1y + c_1z = 0, \tag{17}$$

$$a_2x + b_2y + c_2z = 0, \tag{18}$$

$$a_3x + b_3y + c_3z = 0, \tag{19}$$

when $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \tag{20}$$

but some minor of Δ is not zero, say $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \tag{21}$$

To prove that (17), (18), (19) have an infinite number of solutions, substitute in (19) the values

$$x = \frac{\begin{vmatrix} -c_1z & b_1 \\ -c_2z & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & -c_1z \\ a_2 & -c_2z \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

* A homogeneous equation is one in which all the terms are of the same degree in the unknowns.

which satisfy (17) and (18) when condition (21) is fulfilled. This substitution gives, after clearing of fractions,

$$-za_3\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} - zb_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + zc_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which, by (20), vanishes whatever value be assigned to z. Hence, z can take an infinite number of values, each of which with the corresponding x and y satisfies (17), (18), and (19).

147. Systems of equations containing more unknowns than equations. Consider first the single equation

$$3x + 5y - 6 = 0 \tag{1}$$

with two unknowns. It is manifest from our work on loci of equations (Art. 38) that there are an infinite number of pairs of values of x and y which satisfy this equation.

Consider next the two equations,

$$3x - 4y - 2z + 1 = 0, \tag{2}$$

$$4x + 3y - z - 6 = 0 \tag{3}$$

with three unknowns.

We may solve (2) and (3) for x and y in terms of z. This gives

$$x = \frac{\begin{vmatrix} 2z-1 & -4 \\ z+6 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -4 \\ 4 & 3 \end{vmatrix}} = \frac{10z+21}{25}, \tag{4}$$

$$y = \frac{\begin{vmatrix} 3 & 2z-1 \\ 4 & z+6 \end{vmatrix}}{\begin{vmatrix} 3 & -4 \\ 4 & 3 \end{vmatrix}} = \frac{-5z+22}{25}. \tag{5}$$

Any value assigned to z and the corresponding x and y obtained from (4) and (5) satisfy (2) and (3). Hence, the system has an infinite number of solutions.

The main point to be brought out by these illustrations is that, in general, from n equations containing more than n unknowns, we may solve (Art. 146) for some selected n of the unknowns in

terms of the remaining unknowns. We are then at liberty to assign any values to these remaining unknowns, and thus obtain an infinite number of solutions. The problem in the exceptional cases in which it is impossible to solve for a selected set of n unknowns is too complicated to be treated here.

148. Systems of equations containing fewer unknowns than equations. Consider the equations

$$a_1x + b_1y + c_1 = 0, \tag{1}$$

$$a_2x + b_2y + c_2 = 0, \tag{2}$$

$$a_3x + b_3y + c_3 = 0. \tag{3}$$

In order that these three equations may be consistent,* it is necessary that

$$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

which satisfy (1) and (2) shall also satisfy (3). This requires that

$$-a_3\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} - b_3\frac{\begin{vmatrix} a_1 & c \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + c_3 = 0.$$

Clearing of fractions, and interchanging columns in $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, we obtain

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \quad \text{(Art. 144)} \tag{4}$$

* Two or more equations are *consistent* (Art. 39) when they have a common solution.

as a condition to be satisfied in order that equations (1), (2), and (3) be consistent. Stated in words, in order that three linear equations in two unknowns have a common solution, it is necessary that the determinant formed of the coefficients of the unknowns and of the known terms vanish.

The method used for three linear equations in two unknowns can clearly be extended to any number n of linear equations in $n-1$ unknowns. It results that *the determinant formed of the coefficients of the unknowns and of the known terms must vanish in order that the n equations in $n-1$ unknowns have a common solution.*

While the vanishing of this determinant is a necessary condition for the existence of a common root, it is not a sufficient condition as is shown by the following example.

Take the system of equations

$$x+y-4=0, \tag{5}$$

$$2x+2y+5=0, \tag{6}$$

$$x+y-6=0. \tag{7}$$

Here,
$$\begin{vmatrix} 1 & 1 & -4 \\ 2 & 2 & 5 \\ 1 & 1 & -6 \end{vmatrix} = 0,$$

but any two of the equations are inconsistent.

In establishing the above necessary condition, we assumed that two of the equations have a solution. This condition is satisfied by no two of equations (5), (6), (7).

149. Common roots of quadratic and higher degree equations in one unknown. Consider first the system,

$$ax+b=0, \quad a \neq 0, \tag{1}$$

$$a'x^2+b'x+c'=0, \quad a' \neq 0, \tag{2}$$

consisting of one linear and one quadratic equation. In order that (1) and (2) have a common root, it is necessary and sufficient that the solution $x=-\frac{b}{a}$ which satisfies (1) shall also satisfy (2). This requires that

$$\frac{a'b^2}{a^2} - \frac{b'b}{a} + c' = 0, \tag{3}$$

or

$$a'b^2 - abb' + a^2c' = 0. \tag{4}$$

The relation (4) among the coefficients is the condition that (1) and (2) have a common root.

The left-hand member of (4) may be put into determinant form as follows: Multiply (1) by x, and the resulting equation in combination with (1) and (2) gives the system

$$ax + b = 0, \tag{1}$$

$$ax^2 + bx = 0, \tag{5}$$

$$a'x^2 + b'x + c' = 0, \tag{2}$$

which should be thought of as linear equations in two unknowns, x and x^2. From Art. 148, it is necessary that

$$\begin{vmatrix} 0 & a & b \\ a & b & 0 \\ a' & b' & c' \end{vmatrix} = 0, \tag{6}$$

in order that (1), (2), and (5) have a common root. But (6) is merely (4) written in determinant form, as can be easily verified.

Consider next the equations

$$ax^2 + bx + c = 0, \qquad a \neq 0, \tag{7}$$

$$a'x^3 + b'x^2 + c'x + d' = 0, \qquad a' \neq 0. \tag{8}$$

If we multiply (7) by x and by x^2, and (8) by x, to form additional equations, we obtain

$$ax^2 + bx + c = 0,$$

$$ax^3 + bx^2 + cx = 0,$$

$$ax^4 + bx^3 + cx^2 = 0,$$

$$a'x^3 + b'x^2 + c'x + d' = 0,$$

$$a'x^4 + b'x^3 + c'x^2 + d'x = 0,$$

which can be treated as linear equations in four unknowns, x, x^2, x^3, and x^4. From Art. 148, it is necessary that

$$\begin{vmatrix} 0 & 0 & a & b & c \\ 0 & a & b & c & 0 \\ a & b & c & 0 & 0 \\ 0 & a' & b' & c' & d' \\ a' & b' & c' & d' & 0 \end{vmatrix} = 0, \tag{9}$$

in order that (7) and (8) have a common root. Moreover, when condition (9) is satisfied, (7) and (8) have a common root, but the proof* is beyond the scope of this book.

A relation such as (6) or (9) which results from eliminating the unknowns from systems of equations is often called the *eliminant* of the equations.

The method of elimination employed in the last two examples is called Sylvester's method of elimination. It consists in forming from the given equations additional equations by multiplying the given equations by successive powers of x until we have one more equation than powers of x. These powers of x are then treated as distinct unknowns and the eliminant is obtained as in Art. 148.

EXERCISES

Solve by using determinants.

1. $4x - 3y = 5,$
$8x + y = 17.$

2. $3x + 4y - 2z = 5,$
$4x - 3y + 8z = -4,$
$2x + 8y - 3z - 5 = 0.$

3. $\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 1,$
$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 1,$
$\frac{1}{4x} + \frac{1}{4y} + \frac{1}{2z} = 1.$

4. $3x + 2y + 4z - w - 13 = 0,$
$5x + y - z + 2w - 9 = 0,$
$2x + 3y - 7z + 3w - 14 = 0,$
$4x - 4y + 3z - 5w - 4 = 0.$

5. Find a value of k such that

$$kx - 3y - 5 = 0,$$
$$8x + y - 17 = 0,$$
$$kx + 2y - 10 = 0$$

are consistent equations. Can k take more than one value?

* For proof, see Bocher's *Introduction to Higher Algebra*, pp. 200–202.

6. Discuss the number of values of x, y, z which satisfy

$$\begin{aligned} x + 3y - z &= 0, \\ -2y + z &= 0, \\ 5x + y + 2z &= 0, \end{aligned}$$

and find the ratios $x : y : z$ of corresponding values (apart from $x = y = z = 0$).

7. Eliminate k from the equations

$$\begin{aligned} k^2x - 2kx^2 + 1 &= 0, \\ x^2 + k - 3kx &= 0 \end{aligned}$$

by Sylvester's method.

8. Find the eliminant of $ax^2 + bx + c = 0$ and $x^3 = 1$.

9. Are the equations $x^2 + 3x + 2 = 0$ and $x^3 + 8x^2 + 9x + 2 = 0$ consistent?

10. Determine b so that

$$\begin{aligned} 3x^2 - 8x - 3 &= 0, \\ x^3 - bx^2 - x - 6 &= 0 \end{aligned}$$

have a common root.

11. By Art. 148, it is shown that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ is a necessary condition that the two equations

$$\begin{aligned} a_1x + b_1 &= 0, \quad (a_1 \neq 0) \\ a_2x + b_2 &= 0 \quad (a_2 \neq 0) \end{aligned}$$

be consistent. Show that this condition is also sufficient for this special case.

CHAPTER XIX

LIMITS

150. Definition. *If a is a constant and x is a variable which assumes in order a given sequence of values in such a manner that $|a-x|$* becomes and remains less than any assigned number d $(d>0)$, then x is said to approach a as a limit.*

We have had many illustrations of limits in elementary mathematics. Thus, in geometry the area of a circle is considered as the limiting value of the area of the inscribed regular polygon as the number of sides is indefinitely increased. Here the terms of the "given sequence of values" are the areas of the inscribed polygons as the number of sides is increased. Again, as we annex 3's to the decimal .3333 ···, its value runs through the sequence of numbers .3, .33, .333, etc., which can be made to approach as near to $\frac{1}{3}$ as we please. In the geometrical progression

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots,$$

S_n, the sum of the first n terms, runs through the sequence

$$1, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{15}{8}, \cdots,$$

and approaches the limiting value 2.

The essence of the definition of a limit lies in the words "becomes and remains less." For example, if x runs through the sequence of values $\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \cdots$,

the difference $|1-x|$ takes on the values

$$\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{1}{4}, \tfrac{5}{4} \cdots$$

and becomes less than any assigned number but it does not remain so. In this case we cannot say that x approaches 1 as a limit.

To indicate that x approaches a as a limit, we use the notation

$$x \rightarrow a, \text{ or } \lim x = a.$$

* The notation $|a-x|$ means the absolute value of $a-x$. (Cf. footnote, p. 87.)

151. Infinitesimals. A very important class of variables which are assumed to run through a sequence of values consists of those which have the limit zero. They are called **infinitesimals.** The area between a circle and the inscribed regular polygon as the number of sides increases, the weight of the air in the receiver of a perfectly working air pump, and the difference $2 - S_n$, where S_n is the sum of the first n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$, are examples of infinitesimals.

THEOREM. *If* $u \to 0$ *and* $v \to 0$, *and* X *and* Y *are always numerically less than some positive constant* k, *then* $Xu + Yv \to 0$. In other words, *if* u *and* v *are infinitesimals, then* $Xu + Yv$ *is an infinitesimal.*

Let d be any positive number however small. Since $\lim u = 0$, and $\lim v = 0$, $|u|$ and $|v|$ will ultimately become and remain less than $\frac{d}{2k}$. For these values of u and v, we have

$$|Xu| < \frac{|X|d}{2k},$$

$$|Yv| < \frac{|Y|d}{2k},$$

whence, $$|Xu| + |Yv| < \frac{(|X| + |Y|)d}{2k}.$$

By hypothesis, $$|X| + |Y| < 2k,$$

hence, $$|Xu| + |Yv| < d.$$

But the absolute value of a sum is never greater than the sum of the absolute values of the numbers. (See Art. (56) Exercise 11.)

Hence, $$|Xu + Yv| \leqq |Xu| + |Yv|$$

and $$|Xu + Yv| < d.$$

Since d may be chosen as small as we please,

$$Xu + Yv \to 0.$$

This theorem may be extended to any number of variables.

COROLLARY. *If* $u \to 0$ *and* $v \to 0$, *and* C *is a constant, then*

$$Xu + Yv + C \to C.$$

EXAMPLES: If $u \to 0$ and $v \to 0$

then (1) $7u + 3v \to 0$,

(2) $a + b - u - v \to a + b$,

(3) $ab - (a - u)v - bu \to ab$.

152. Theorems concerning limits. The following theorems follow directly from the theorem of Art. 151.

THEOREM I. *The limit of the sum of two variables is the sum of their limits.*

Let the variables be x and y, and let

$$\lim x = a, \quad \lim y = b.$$

Then, $$x = a - u, \quad y = b - v,$$

where $$u \to 0, \quad v \to 0.$$

Adding, we have $x + y = a + b - (u + v)$.

From the corollary of Art. 151, $a + b - (u + v) \to 0$,

or $$\lim (x + y) = a + b = \lim x + \lim y.$$

COROLLARY I. *The limit of the sum of any finite number of variables is the sum of their limits.*

COROLLARY II. *The limit of the difference of two variables is the difference of their limits.*

THEOREM II. *The limit of the product of two variables is the product of their limits.*

Using the notation of Theorem I,

$$xy = (a - u)(b - v) = ab - [(a - u)v + bu].$$

From the theorem of Art. 151,

$$(a - u)v + bu \to 0.$$

Hence, $$\lim xy = ab = \lim x \lim y.$$

COROLLARY I. *The limit of the product of any finite number of variables is the product of their limits.*

COROLLARY II. *If n is a positive integer,*

$$\lim x^n = a^n = (\lim x)^n.$$

COROLLARY III. *If c is any constant,*

$$\lim cx = c \lim x.$$

153. Both numerator and denominator with limit zero. If both the numerator and the denominator of a fraction $\frac{x}{y}$ approach the limit zero, we have a rather curious result, as is shown by the cases which occur in the following example.

In the fraction $\frac{x}{y}$ let y approach 0 through the sequence of values

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n} \cdots.$$

Let x approach 0 through one of the four sequences:

(a) $$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \cdots, \frac{1}{4^n}, \cdots.$$

(b) $$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \cdots, \frac{1}{\sqrt{2^n}}, \cdots.$$

(c) $$\frac{k}{2}, \frac{k}{2^2}, \frac{k}{2^3}, \cdots, \frac{k}{2^n}, \cdots. \quad (k = \text{any constant.})$$

(d) $$\frac{1}{2}, -\frac{1}{2^2}, \frac{1}{2^3}, -\frac{1}{2^4}, \cdots, \pm\frac{1}{2^n}, \cdots.$$

Case (a). We have here $\lim \frac{x}{y} = \lim_{n \to \infty} \frac{\frac{1}{4^n}}{\frac{1}{2^n}} = \lim_{n \to \infty} \frac{1}{2^n} = 0.$

(See Art. 154.)

Case (b). Here $\frac{x}{y}$ passes through the sequences of values

$$\sqrt{2}, \sqrt{2^2}, \sqrt{2^3}, \cdots, \sqrt{2^n}, \cdots$$

which increases without limit.

Case (c). In this case $\lim \frac{x}{y} = k$.

Case (d). Here $\frac{x}{y}$ takes alternately the values $+1$ or -1 and approaches no limit.

We see then that if x and y both approach 0 as a limit, their ratio may approach any number whatever including 0, may increase without limit, or may oscillate between two fixed numbers.

154. Infinity. If the numerator of the fraction is constant, or has the limit $a (a \neq 0)$, while the denominator has the limit 0, then $\frac{x}{y}$ increases without limit and is said to become **infinite.** This is usually expressed by writing $\lim_{y \to 0} \frac{x}{y} = \infty$.

It is not, however, to be inferred that infinity is a limit. The variable $\frac{x}{y}$ in the case just given approaches no limit. If z is a variable which increases without limit, the various expressions "$\lim z = \infty$," "$z \to \infty$," "$z = \infty$," should not be read "z approaches infinity" or "z equals infinity," but "z becomes infinite," "z increases without limit." Infinity is not a number in the sense in which we are using the term.

THEOREM I. $$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Let d be any assigned small positive number. Let $n \geqq \frac{x}{d}$ where x is any number greater than 1. Then $\frac{1}{n} \leqq \frac{d}{x} < d$. That is, $\frac{1}{n}$ becomes and remains less than any assigned number.

THEOREM II. If $|r| < 1$, $\lim_{n \to \infty} r^n = 0$.

Since $|r| < 1$, it can be written in the form $|r| = \frac{1}{1+h}$ where h is positive. Hence,

$$|r^n| = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + \text{positive terms}}.$$

(By Binomial Theorem.)

Therefore, $$|r^n| < \frac{1}{1+nh} < \frac{1}{nh}.$$

By Theorem I of the present article and Corollary III, Art. 152,

$$\lim_{n \to \infty} \frac{1}{nh} = 0.$$

Hence, $$\lim_{n \to \infty} |r^n| = 0.$$

Since $r^n = \pm |r^n|$, we have

$$\lim_{n \to \infty} r^n = 0.$$

COROLLARY. *If* $|r|<1$, $\lim_{n\to\infty} \frac{ar^n}{1-r}=0.$

Exercise. Let y approach 0 through the sequence

$$0.1,\ 0.01,\ 0.001,\ \cdots.$$

Show that the fraction $\frac{x}{y}$ may be made to approach any number as a limit, may increase without limit, or may oscillate between two numbers.

155. Limiting value of a function. Let $f(x)$ represent any function of x. If x runs through any sequence of values and approaches a limit a and at the same time $f(x)$ takes on a sequence of corresponding values such that

$$\lim f(x)=A,$$

we may abbreviate and write

$$\lim_{x\to a} f(x)=A,$$

which reads, "As x approaches a through any sequence of values, $f(x)$ approaches the limit A"; or, more briefly, "The limit of $f(x)$, when x approaches a, is A."

If $$f(a)=\lim_{x\to a} f(x),$$

the function is said to be **continuous** for $x=a$.

156. Indeterminate forms. To find the value of the fraction $\frac{x^2+x-2}{x-1}$ when $x=2$, we substitute and find the value to be 4. But when $x=1$, by substitution we find $\frac{0}{0}$, a meaningless symbol.

We may write $$\frac{x^2+x-2}{x-1}=x+2,$$

but since division by zero is excluded from our operations, this simplification does not hold for $x=1$. But for every other value of x, however near to 1, the division is possible. Hence, letting x approach 1, we have

$$\lim_{x\to 1}\frac{x^2+x-2}{x-1}=\lim_{x\to 1}(x+2)=3.$$

Although substitution of $x=1$ in $\dfrac{x^2+x-2}{x-2}$ gives us a meaningless symbol, it is convenient to assign a value to the fraction. When $x=1$, we *define* $\dfrac{x^2+x-2}{x-1}$ to be $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x-1}=3$. Giving this value to the fraction makes $\dfrac{x^2+x-2}{x-1}=x+2$ true for all values of x. In general, if $f(x)$ is a fraction which for $x=a$ takes the form $\dfrac{0}{0}$, we define $f(a)$ to be $\lim\limits_{x\to a} f(x)$.

The student should note that this is not a *necessary* definition of $f(a)$, but merely a convenient one. The convenience arises from the fact that with such a definition of $f(a)$, the function becomes continuous at $x=a$.

We wish sometimes to find the limit of the value of a function as the variable increases without limit. The following example illustrates the method.

Find the limit of $\dfrac{x^2+2}{3x^2+2x-1}$ for $x\to\infty$.

By the theorems on limits this will take the meaningless form $\dfrac{\infty}{\infty}$, but dividing numerator and denominator by x^2, we can write the fraction as

$$\frac{1+\dfrac{2}{x^2}}{3+\dfrac{2}{x}-\dfrac{1}{x^2}},$$

and since $\dfrac{2}{x^2}$, $\dfrac{2}{x}$, $\dfrac{1}{x^2}$ are infinitesimals by Theorem I, Art. 154, we have $\dfrac{1}{3}$ for the limit of the fraction.

The symbols $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$ are called indeterminate forms. Among other such forms which may arise are $0\cdot\infty$ and $\infty-\infty$, but the expressions which give rise to these forms may be reduced to the form $\dfrac{0}{0}$, as shown in the following examples.

EXAMPLE 1: $(x^2+x-2)\cdot\dfrac{1}{x-1}$ takes the form $0\cdot\infty$ when $x=1$. For any other value of x we may write

$$(x^2+x-2)\cdot\frac{1}{x-1}=\frac{x^2+x-2}{x-1}.$$

Hence, $$\lim_{x\to 1}(x^2+x-2)\cdot\frac{1}{x-1}=\lim_{x\to 1}\left(\frac{x^2+x-2}{x-1}\right)=3.$$

EXAMPLE 2: $\dfrac{x-1}{x^2-9}-\dfrac{x^2+x-6}{x(x^2-9)}$ takes the form $\infty-\infty$ when $x=3$. For any other value of x, we may write

$$\frac{x-1}{x^2-9}-\frac{x^2+x-6}{x(x^2-9)}=\frac{-2(x-3)}{x(x^2-9)}=\frac{-2}{x(x+3)}.$$

Hence, $$\lim_{x\to 3}\left(\frac{x-1}{x^2-9}-\frac{x^2+x-6}{x(x^2-9)}\right)=\lim_{x\to 3}\frac{-2}{x(x+3)}=-\frac{1}{9}.$$

EXERCISES

What values should be given to the following expressions in order to make them continuous for values of x indicated?

1. $\dfrac{x^3-1}{x-1}$ when $x=1$.

2. $\dfrac{x^2-5x+6}{x^2-4}$ when $x=2$.

3. $\dfrac{x-3}{\sqrt{x}-\sqrt{3}}$ when $x=3$.

4. $\dfrac{1-x^5}{1-x^2}$ when $x=1$.

5. $\dfrac{x^3+a^3}{x^2-a^2}$ when $x=-a$.

6. $(x^2-4)\cdot\dfrac{1}{x-2}$ when $x=2$.

7. $\dfrac{1}{x}-\dfrac{2}{x(x+2)}$ when $x=0$.

8. $\dfrac{x^2}{(x-1)^2}-\dfrac{2x-1}{(x-1)^2}$ when $x=1$.

9. $\dfrac{x^2+x+1}{x+1}$ when $x=-1$.

As x increases without limit find the limits of the following fractions.

10. $\dfrac{x+1}{x}$.

11. $\dfrac{x}{x^2+1}$.

12. $\dfrac{x^3-3x}{2x^3+5x^2}$.

CHAPTER XX

INFINITE SERIES

157. Definition. Let $u_1, u_2, \cdots, u_n, \cdots$ be any unending sequence of real numbers positive or negative. The expression

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots,$$

when the terms are formed according to some law of succession, is called an *infinite series.*

In the discussion of geometrical and harmonical progressions we have met such series, for example,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots,$$

and

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots.$$

158. Convergence and divergence. In the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

let S_n represent the sum of the first n terms, that is,

$$\begin{aligned} S_1 &= u_1, \\ S_2 &= u_1 + u_2, \\ S_3 &= u_1 + u_2 + u_3, \\ &\cdots\cdots\cdots \\ S_n &= u_1 + u_2 + \cdots + u_n, \\ &\cdots\cdots\cdots \end{aligned}$$

For example, in the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

given in Art. 157, we have

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1 + \tfrac{1}{2} = \tfrac{3}{2}, \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4}, \\ &\cdots\cdots\cdots \\ S_n &= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}. \end{aligned}$$

In the series $1+2+3+4+\cdots$, we have

$$S_1 = 1,$$
$$S_2 = 1+2 = 3,$$
$$S_3 = 1+2+3 = 6,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_n = 1+2+3+\cdots+n = \frac{n}{2}(1+n).$$

In the series $1-1+1-1+1-\cdots$, we have

$$S_1 = 1,$$
$$S_2 = 0,$$
$$S_3 = 1,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_n = 1 \text{ or } 0.$$

These three examples illustrate three cases which may occur.

I. S_n approaches a limit as n increases without limit. In the first example above, S_n is never greater than 2, no matter how large a number n represents and approaches 2 as a limit, when n increases without limit.

II. S_n is numerically larger than any assigned number for a sufficiently large value of n.

This case is illustrated in the second problem.

III. S_n remains finite but does not approach a limit as n increases without limit.

This case is illustrated in the third series, where S_n may have either of the values 0 or 1, according as n is even or odd.

Series which come under Case I are called **convergent** series and are by far the most important. Series which are included in Cases II and III are called **divergent** series. We have then the

DEFINITION. *When in an infinite series the sum of the first n terms approaches a limit as n increases without limit, the series is said to be convergent. Otherwise it is divergent.*

The *limit of the sum* of n terms of a convergent series, written $\lim S_n$, is often called the *sum** of the series. In connection

* The word "sum" is here used in a purely conventional sense. It is not to be understood as the sum of an infinite number of terms, but as the limit of the sum of n terms as n increases without limit.

with convergent series we shall also use the expression "limiting value of the series" to mean $\lim S_n$.

Many important mathematical investigations and the solution of many practical problems require the use of infinite series and depend upon the question of convergence. The central problem of this chapter will be the problem of deciding whether a given series is convergent or divergent. It will be found convenient to take up first those series in which the terms are positive numbers.

SERIES WITH POSITIVE TERMS

159. Fundamental assumption. *An infinite series of positive terms is convergent if S_n is always less than some definite number, however great n may be.**

Let K be a number such that $S_n < K$ for all values of n. Since the series contains positive terms only, S_n is a variable which increases as n increases. Since it can never attain so great a value as K, we assume that there is some number less than K which S_n approaches as a limit.

To illustrate this assumption, consider the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots,$$

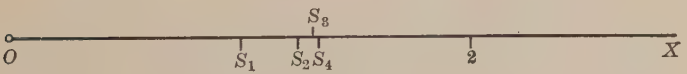

FIG. 33.

and take points on the line OX (Fig. 33), to represent S_1, S_2, S_3, ··· so that the measure of OS_1, is S_1, of OS_2, is S_2, etc.

$$S_1 = 1,$$

$$S_2 = 1 + \frac{1}{2^2} = 1.2500,$$

$$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^3} = 1.2870,$$

$$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} = 1.2909,$$

. . . . . .

* For proof see Pierpont's *Theory of Functions of a Real Variable*, Art. 109. Fine's *College Algebra*, p. 59, Art. 192.

We can show that the sum of n terms of this series is less than 2. (See Art. 160, Example 1.) Hence, according to the assumption of this section, there is some point not farther to the right than 2 which S_n approaches as a limit when $n \to \infty$.

An analogous assumption exists for a series all of whose terms are negative. An infinite series of negative terms is convergent, if S_n is always algebraically greater than some definite number, however great n may be.

EXERCISES

1. It can be shown that the sum of the series,

$$1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \cdots + \frac{1}{(n-1)\,!} + \cdots,^*$$

is always less than 3. Illustrate the assumption of this section graphically by means of this series.

2. Illustrate graphically the assumption for a series of negative terms by means of the series,

$$-1 - \frac{1}{2^2} - \frac{1}{3^3} - \frac{1}{4^4} - \cdots.$$

160. Comparison test for convergence. *Consider the series of positive terms*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots.$$

If from some term on, the terms of this series are equal to or less than the corresponding terms of a known convergent series,

$$v_1 + v_2 + v_3 + \cdots + v_n + \cdots$$

of positive terms, then the u series is convergent.

Let $$S_n = u_1 + u_2 + \cdots + u_n,$$

and $$S_n' = v_1 + v_2 + \cdots + v_n.$$

Suppose that after the kth term

$$u_{k+1} \leqq v_{k+1},\ u_{k+2} \leqq v_{k+2},\ \cdots,\ u_n \leqq v_n,\ \cdots,$$

then, if $$S_k = u_1 + u_2 + \cdots + u_k,$$

and $$S_k' = v_1 + v_2 + \cdots + v_k,$$

we have $$S_n - S_k \leqq S_n' - S_k',$$

or $$S_n \leqq S_n' - S_k' + S_k < \lim S_n' - S_k' + S_k.$$

* For meaning of 1 !, 2 !, 3 !, etc., see Art. 59.

Since by hypothesis $\lim S_n'$ exists, it follows that S_n is always less than a definite number, and by the assumption of Art. 159 the series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is convergent.

EXAMPLE 1: Prove the series

$$2 + 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{(n-1)^{n-1}} + \cdots$$

to be convergent.

SOLUTION: For purposes of comparison take as the v series the geometrical progression

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots,$$

which we have shown to be convergent (Art. 157). Write the given series under the comparison series:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots,$$

$$2 + 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{(n-1)^{n-1}} + \cdots.$$

After the third term, each term in the second series is less than the corresponding term just above it. That this is true for every term after the third is shown by examining the two nth terms. If $n > 3$, then

$$\frac{1}{(n-1)^{n-1}} < \frac{1}{2^{n-1}}.$$

Beginning with the fourth term, the sum of n terms of the first series is always less than $\frac{1}{4}$. Hence, the sum of the second series can never exceed $2 + 1 + \frac{1}{2^2} + \frac{1}{4} = 3\frac{1}{2}$. In comparing two series it is not sufficient to compare a few terms at the beginning of the series. The nth terms should be compared.

EXAMPLE 2: Test for convergence the series

$$\frac{3 \cdot 3^3}{1} + \frac{2 \cdot 3^2}{1} + \frac{3}{1} + 1 + \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \cdots.$$

If we neglect the first three terms of this series and prove the remainder to be convergent, the given series must converge. For, if the series beginning with the fourth term has a sum, the sum of the entire series will be the sum of terms after the third plus $3 \cdot 3^3 + 2 \cdot 3^2 + 3 = 102$. Beginning then with the fourth term and comparing with the series,

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \cdots$$

which is known to converge to $\frac{3}{2}$ (Art. 73), we have

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n-1}} + \cdots,$$

and

$$1 + \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \cdots + \frac{1}{(n-1) \cdot 3^{n-1}} + \cdots.$$

Each term in the second row is equal to or less than the corresponding term in the first. Hence, the second series converges to some number not greater than $\frac{3}{2}$ and the sum of the entire series in question is not greater than 103.5.

In testing for convergence it is often convenient to omit a finite number of terms as in the above example. That this is always justified is shown by the following

THEOREM. *The convergence of a series is not affected by neglecting a finite number of terms.*

For the sum of the terms neglected is a definite number which added to the sum of the new series gives a definite number for the sum of the entire series.

EXERCISES

Test the following series for convergence:

1. $1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}+\frac{1}{4\cdot 2^3}+\cdots.$

2. $1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots.$

3. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots.$

SOLUTION: Write S_n in the form

$$S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

Hence, $$\lim_{n\to\infty} S_n=1,$$

4. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots.$

5. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.$

6. $1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots.$

7. $1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$, where $p\geqq 2$.

8. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\cdots$, where $x<1$.

9. $1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$, where $p>1$.

SOLUTION: Write down the inequalities,

$$\frac{1}{2^p}+\frac{1}{3^p}\quad\frac{2}{2^p}=\frac{1}{2^{p-1}},$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}=\frac{1}{4^{p-1}},$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\cdots+\frac{1}{15^p}<\frac{8}{8^p}=\frac{1}{8^{p-1}}.$$

. . . . . .

Add the members of the inequalities, thus

$$\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots<\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\cdots.$$

The right-hand side of this inequality is a geometrical progression whose ratio is $\frac{1}{2^{p-1}}$, which is <1 when $p>1$. Hence, the series is convergent. This is a useful series for testing other series.

10. $1+\frac{x}{2^{\frac{3}{2}}}+\frac{x^2}{3^{\frac{3}{2}}}+\frac{x^3}{4^{\frac{3}{2}}}+\cdots$, where $x\lessgtr 1$.

161. Comparison test for divergence. *Given the series of positive terms,*

$$u_1+u_2+\cdots+u_n+\cdots.$$

If from some term on, the terms of the series are equal to or greater than the corresponding terms of a known divergent series of positive terms,

$$v_1+v_2+\cdots+v_n+\cdots,$$

then the given series is divergent.

Let $$S_n=u_1+u_2+\cdots+u_n,$$

and $$S_n'=v_1+v_2+\cdots+v_n.$$

After the kth term, suppose

$$u_{k+1}\gtreqless v_{k+1},\ u_{k+2}\gtreqless v_{k+2},\ \cdots,\ u_n\gtreqless v_n,$$

then $$S_n-S_k\gtreqless S_n'-S_k',$$

or $$S_n\gtreqless S_n'-S_k'+S_k.$$

By hypothesis S_n' increases without limit as n increases. Hence, if n is made large enough, S_n will exceed any given number, and the u series is divergent.

A useful comparison series for divergence is the harmonical series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots,$$

which can be shown to be divergent by means of the inequalities:

$$1+\tfrac{1}{2}>1,$$

$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2},$$

$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}=\tfrac{1}{2},$$

$$\tfrac{1}{9}+\tfrac{1}{10}+\cdots+\tfrac{1}{16}>\tfrac{1}{2},$$

. . . . . .

Adding members of the inequalities, we have

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\cdots.$$

But the right-hand member of this inequality can be made as large as we please. The series in question is therefore divergent.

Any geometrical progression $a+ar+ar^2+\cdots$ in which the ratio is greater than 1 can be shown to be divergent by comparison with the series $a+a+a+\cdots$; and such a geometrical progression is often useful as a comparison series in testing for divergence.

EXERCISES

Prove the following series divergent.

1. $1+2+3+4+\cdots$.

2. $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\cdots$.

3. $1+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\dfrac{1}{4^p}+\cdots$ where p is positive and less than 1.

4. $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{6}+\dfrac{1}{8}+\cdots$.

5. $1+\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\cdots$.

162. Summary of standard test series. When any new series has been shown to be convergent or divergent, we evidently increase our supply of series for comparison purposes, but a few standard series are so useful as to deserve special mention.

For convergence:

1. $a+ar+ar^2+\cdots+ar^{n-1}+\cdots(r<1)$.

2. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}+\cdots$.

3. $1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots(p>1)$.

For divergence:

1. $a+ar+ar^2+\cdots+ar^{n-1}+\cdots(r\geqq 1)$.

2. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$.

163. Ratio test for convergence and divergence. Another important test for the convergence or divergence of an infinite series is the so-called ratio test.

THEOREM. *If, as n increases beyond bound, the ratio $\frac{u_{n+1}}{u_n}$ approaches a limit λ, the series of positive terms*

$$u_1+u_2+\cdots+u_n+\cdots$$

is convergent if $\lambda<1$ and divergent if $\lambda>1$. If $\lambda=1$, this test fails.

1. $\lambda<1$. Since $\lim\frac{u_{n+1}}{u_n}=\lambda$, we can make $\frac{u_{n+1}}{u_n}$ differ from λ

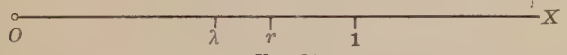

FIG. 34.

by as small a number as we please. Hence, if we plot values of $\frac{u_{n+1}}{u_n}$ on the line OX, as n increases the points representing $\frac{u_{n+1}}{u_n}$ will concentrate about the point λ. If n is taken large enough, they will lie to the left of the point r, where $\lambda<r<1$. For these values of n, we have

$$\frac{u_{n+1}}{u_n}<r, \qquad u_{n+1}<ru_n,$$

$$\frac{u_{n+2}}{u_{n+1}}<r, \quad u_{n+2}<ru_{n+1}<r^2u_n,$$

$$\frac{u_{n+3}}{u_{n+2}}<r, \qquad u_{n+3}<r^3u_n,$$

. . . . . .

Since $r<1$, the series

$$ru_n+r^2u_n+r^3u_n+\cdots=u_n(r+r^2+r^3+\cdots)$$

is convergent. But each term of the series

$$u_{n+1}+u_{n+2}+u_{n+3}+\cdots$$

is less than the corresponding term of the ru series. Hence, by Art. 160, the series $u_1 + u_2 + \cdots u_n + \cdots$ is convergent.

2. $\lambda > 1$. In this case the points representing $\frac{u_{n+1}}{u_n}$ will ultimately concentrate about the point λ on the line OX, and if n is

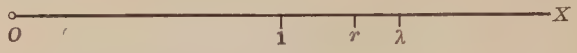

FIG. 35.

large enough, they will lie to the right of the point r, where $1 < r < \lambda$. Then

$$\frac{u_{n+1}}{u_n} > r,$$

or $$u_{n+1} > ru_n;$$

and $$u_{n+2} > r^2u_n,$$

$$u_{n+3} > r^3u_n,$$

. . . .

Therefore, since the series

$$ru_n + r^2u_n + r^3u_n + \cdots$$

is divergent for $r > 1$,

the series $$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is divergent (Art. 161).

3. $\lambda = 1$. If $\lim \frac{u_{n+1}}{u_n} = 1$, this test fails. This is illustrated in the two series,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots,$$

and $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

The first has been shown to be convergent (Art. 160), the second divergent (Art. 161), but for each $\lim \frac{u_{n+1}}{u_n} = 1$.

EXAMPLE 1: Consider the series

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots.$$

Here, $u_{n+1}=\dfrac{n+1}{2^{n+1}}$, $u_n=\dfrac{n}{2^n}$,

$$\frac{u_{n+1}}{u_n}=\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}=\frac{n+1}{2n}=\frac{n}{2n}+\frac{1}{2n}=\frac{1}{2}+\frac{1}{2n}.$$

$$\lim\frac{u_{n+1}}{u_n}=\lim\left(\frac{1}{2}+\frac{1}{2n}\right)=\frac{1}{2}.$$

Hence, the series is convergent.

EXAMPLE 2: Consider the series

$$\frac{2}{2^2}+\frac{2^2}{3^2}+\frac{2^3}{4^2}+\frac{2^4}{5^2}+\cdots.$$

Here, $u_{n+1}=\dfrac{2^{n+1}}{(n+2)^2}$, $u_n=\dfrac{2^n}{(n+1)^2}$,

$$\frac{u_{n+1}}{u_n}=\frac{2^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{2^n}=2\left(\frac{n+1}{n+2}\right)^2.$$

$$\lim\frac{u_{n+1}}{u_n}=2.$$

Hence, the series is divergent.

EXERCISES

Apply the ratio test to the following series:

1. $1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\cdots.$

2. $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\dfrac{4}{2^4}+\cdots.$

3. $1+\dfrac{2^2}{2!}+\dfrac{3^2}{3!}+\dfrac{4^2}{4!}+\cdots.$

4. $\dfrac{2!}{100}+\dfrac{3!}{100^2}+\dfrac{4!}{100^3}+\cdots.$

5. $2+\dfrac{2^2}{2}+\dfrac{2^3}{3}+\dfrac{2^4}{4}+\cdots.$

6. $1+\dfrac{2^p}{2!}+\dfrac{3^p}{3!}+\dfrac{4^p}{4!}+\cdots$, where p may have any value.

7. $1+\dfrac{1}{3!}+\dfrac{1}{5!}+\cdots.$

8. $1+2+4+8+\cdots.$

9. $1+\dfrac{2^2}{2!}+\dfrac{3^2}{3!}+\dfrac{4^2}{4!}+\cdots.$

SERIES WITH BOTH POSITIVE AND NEGATIVE TERMS

164. Thus far we have considered only series whose terms are all positive or all negative. The following theorem will throw light on the convergence of series whose terms are not all of the same sign.

Theorem. *An infinite series of real terms which are not all of the same sign is convergent if the series formed by making all the terms positive is convergent.*

After all the minus signs have been changed to plus signs, let the series be $u_1 + u_2 + u_3 + \cdots$.

By hypothesis, this series is convergent and therefore has a limiting value S. The sum of the first n terms of this series is then less than S. Hence, the sum, S_n, of the first n terms of the original series is numerically less than S. Let these n terms consist of p positive and q negative terms. If P_p be the sum of the positive terms and N_q the sum of the negative terms, then

$$S_n = P_p - N_q.$$

But P_p and N_q are always less than S. Hence, by Art. 159, P_p and N_q approach fixed numbers P and N respectively as n increases without limit. Then

$$\lim_{n \to \infty} S_n = P - N, \text{ a definite number,}$$

and the series is convergent.

165. Ratio test extended. The ratio test can readily be extended to series whose terms are not all of the same sign. Since a series of positive terms is convergent if

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} < 1,$$

it follows, from the theorem just proved in Art. 164, that any series is convergent if the numerical value of $\lim_{n \to \infty} \frac{u_{n+1}}{u_n}$ is less than 1. That is, if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

To extend the ratio test for divergence we need the following important

THEOREM. *In any convergent series, the limit of the nth term as n increases without limit is zero.* That is,

$$\lim_{n\to\infty} u_n = 0.$$

For, we may always write $u_n = S_n - S_{n-1}$.

As n increases both S_n and S_{n-1} approach the same limit S; hence (Art. 152), $\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$

While it is necessary that $\lim u_n = 0$ in order that a series be convergent, it is not sufficient. That is, $\lim u_n$ may equal zero and the series be divergent. Exercises 2, 3, 4, 5, p. 228, furnish examples of such series.

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$, the nth term cannot approach zero as a limit, hence, the series is divergent. We may then write the ratio test for *any* infinite series

$$u_1 + u_2 + u_3 + \cdots$$

as follows:

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, *the series converges.*

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$, *the series diverges.*

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 1$, *the test fails.*

EXAMPLE: Test for convergence and divergence the series

$$1 - 2x + 3x^2 - 4x^3 + \cdots.$$

Here, $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)x^n}{nx^{n-1}}\right| = \left|\left(\frac{n+1}{n}\right)x\right|,$$

and $$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x|.$$

Hence, if x lies between $+1$ and -1, the series is convergent. For $|x| > 1$, the series is divergent. The interval between $+1$ and -1 is called the *interval of convergence* of the series, and is represented graphically by the heavy part of the line in Fig. 36. For the points 1 and -1 the test tells us nothing.

FIG. 36.

166. Alternating series. *A series whose terms are alternately positive and negative is convergent if each term is less than the preceding term, and if the nth term approaches zero as a limit when n increases without limit.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n \pm \cdots,$$

where $u_1,\ u_2,\ u_3, \cdots$ are positive,

and $u_2 < u_1, u_3 < u_2, \cdots,$

and where $\lim_{n \to \infty} u_n = 0.$

Let n be an even number. We may then write S_n in the form

$$S_n = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{n-1} - u_n).$$

Since each parenthesis contains a positive number, S_n is positive and increases as n increases. We may also write S_n in the form

$$S_n = u_1 - (u_2 - u_3) - \cdots - (u_{n-2} - u_{n-1}) - u_n.$$

Since the parentheses are again positive,

$$S_n < u_1.$$

By the assumption of Art. 161, as n increases beyond bound, S_n approaches a limit S. But

$$S_{n+1} = S_n + u_{n+1},$$

hence, $\lim S_{n+1} = \lim S_n + \lim u_{n+1}.$

By hypothesis, $\lim u_{n+1} = 0.$

Hence, $\lim S_{n+1} = \lim S_n = S,$

and the series is convergent.

167. Approximate value of a series. In the case of some series, for example, a geometrical progression, we are able to find exactly the limiting value of the sum of n terms as n increases without limit, but with many series we must be content to find an approximation to the limit, say correct to a certain number of decimal places.

EXAMPLE. Calculate

$$1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\frac{1}{9!}+\cdots$$

correct to four decimal places.

$$\begin{aligned} 1&=1.00000 \\ \frac{1}{5!}&=0.00833 \\ \frac{1}{9!}&=0.00000 \\ \hline & 1.00833 \\ -&\ 0.16687 \\ \hline & 0.84146 \end{aligned} \qquad \begin{aligned} -\frac{1}{3!}&=-0.16667 \\ -\frac{1}{7!}&=-0.00020 \\ -\frac{1}{11!}&=-0.00000 \\ \hline & -0.16687 \end{aligned}$$

To four decimal places then the sum is 0.8415. But the question arises as to just where we must stop adding terms. Even if $\frac{1}{11!}$ has no significant figure in the fifth place, we are dropping an infinite number of terms, a number of which when added together may affect the result materially. In the case of an alternating series, this question is easily answered by the

THEOREM. *The sum of the first n terms of an alternating series differs from the sum of the series by less than the* $(n+1)$*th term.*

Let S represent the limit of the sum of the series, S_n the sum of the first n terms, and R_n the remainder. For n, even or odd, we have

$$|R_n| = u_{n+1} - u_{n+2} + u_{n+3} - \cdots.$$

From Art. 166, the sum of this alternating series, whose first term is u_{n+1}, is less than the first term. Hence,

$$|S - S_n| = |R_n| < u_{n+1}.$$

EXERCISES

Test the following series for convergence and divergence.

1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$

2. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots.$

3. $1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots.$

4. $\frac{1}{1+2}+\frac{1}{1+2\cdot 2^2}+\frac{1}{1+3\cdot 2^3}+\cdots.$

5. $\frac{1}{x}-\frac{1}{x+a}+\frac{1}{x+2a}-\frac{1}{x+3a}+\cdots$, $(x>0,\ a>0)$.

6. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots$.

7. $\frac{1}{1.1}+\frac{1}{1.02}+\frac{1}{1.003}+\frac{1}{1.0004}+\cdots$.

8. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$.

9. $\frac{1}{1}+\frac{1\cdot 3}{1\cdot 4}+\frac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7}+\cdots$.

10. $\frac{x}{1}+\frac{x(x+1)}{1\cdot 2}+\frac{x(x+1)(x+2)}{1\cdot 2\cdot 3}+\cdots$, $(x>0)$.

Compute, correct to four decimal places.

11. $1-\frac{1}{3^2}+\frac{1}{5^3}-\frac{1}{7^4}+\cdots$.

12. $\frac{1}{2}-\frac{1}{2^3}\cdot\frac{1}{3!}+\frac{1}{2^5}\cdot\frac{1}{5!}-\frac{1}{2^7}\cdot\frac{1}{7!}+\cdots$.

13. $1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^4}+\cdots$.

14. $\frac{1}{1}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\frac{1}{4}\cdot\frac{1}{3^4}+\cdots$.

Write down the first five terms of the series in which u_n has the following values, and investigate the series for convergence or divergence.

15. $u_n=\frac{1}{3^n}$.

16. $u_n=\frac{1}{n^2}$.

17. $u_n=\frac{2^n}{1+2^{n+1}}$.

18. $u_n=\frac{1}{1+n\sqrt{n}}$.

For what values of x are the following series convergent?

19. $\frac{1}{1+x^2}+\frac{1}{2+2x^2}+\frac{1}{3+3x^2}+\cdots$.

20. $\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{6}+\cdots$.

21. $\frac{1}{2x+1}+\frac{1}{3}\frac{1}{(2x+1)^3}+\frac{1}{5}\frac{1}{(2x+1)^5}+\cdots$.

22. $2x+3\cdot 2x^2+4\cdot 3x^3+\cdots$.

23. $\frac{1}{x+2}+\frac{1}{(x+2)^2}+\frac{1}{(x+2)^3}+\cdots$.

168. Power series. The series

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots,$$

in which a_0, a_1, a_2, $\cdots$ are independent of x, is called a **power series.** Such a series may converge for all values of x, may diverge for all values of x except for $x=0$, or it may converge for some values of x and diverge for others.

THEOREM. *If in the series*

$$a_0 + a_1x + a_2x^2 + \cdots$$

the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ *approaches a limit* r, *then the series converges for* $|x| < \left|\frac{1}{r}\right|$, *and diverges for* $|x| > \left|\frac{1}{r}\right|$.

This result follows directly from Art. 165. Applying the ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_nx^n} = rx.$$

From Art. 165, the series converges if $|rx| < 1$, that is, for

$$|x| < \left|\frac{1}{r}\right|.$$

For $|x| > \left|\frac{1}{r}\right|$ the series diverges. If $r = 0$, the series converges for all values of x. For $x = \frac{1}{r}$, the test fails; the series may or may not be convergent for this value of x.

COROLLARY. *If a power series is convergent for* $x = b$, *it is convergent for every value of* x *numerically less than* b.

EXERCISES

Find the interval of convergence of the following series. Exhibit the results graphically.

1. $\frac{x}{2} + \frac{2^2x^2}{2^2} + \frac{3^2x^3}{2^3} + \cdots$.

SOLUTION: $a_n = \frac{n^2}{2^n}$, $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \left(\frac{n^2}{2n^2} + \frac{2n}{2n^2} + \frac{1}{2n^2}\right) = \frac{1}{2}.$$

The series is therefore convergent for $|x| < 2$ (Fig. 37).

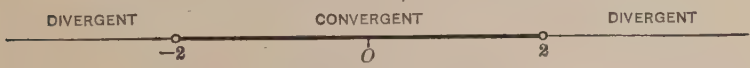

FIG. 37.

2. $1 + 2x + (2x)^2 + (2x)^3 + \cdots$.

3. $1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$.

4. $1 + x + 2!x^2 + 3!x^3 + \cdots (x \neq 0)$.

5. $x + \frac{2x^2}{2!} + \frac{3x^2}{3!} + \cdots$.

6. $1 + 5x + \frac{5^2x^2}{2!} + \frac{5^3x^3}{3!} + \cdots$.

7. $1 - 2x + \frac{2 \cdot 3}{2!}x^2 - \frac{2 \cdot 3 \cdot 4}{3!}x^3 + \frac{2 \cdot 3 \cdot 4 \cdot 5}{4!}x^4 + \cdots$.

By division, expand the following fractions into series and test for convergence.

8. $\frac{1}{1-x}$. **9.** $\frac{1}{1-4x}$. **10.** $\frac{1}{2-3x}$. **11.** $\frac{1}{(1+x)^2}$.

The expansions of $\sin x$, $\cos x$, arc $\sin x$, and arc $\tan x$ are given below. Find the interval of convergence of each of the series.

12. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

13. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

14. $\text{arc sin } x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{x^7}{7} + \cdots$.

15. $\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

16. From the series in Exercise 12, find the value of sin 5° to four decimal places.

HINT: Express 5° in radians.

169. Binomial series. The power series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots.$$

is called the binomial series. If m is a positive integer, the series ends with the $(m+1)$th term and has been shown to be the expansion of $(1+x)^m$. If m is not a positive integer, the series is an infinite series, but it can be shown that it converges towards $(1+x)^m$ when x has any value which makes the series convergent. In other words, it can be shown that for these values of x, the binomial expansion holds for any exponent, integral or fractional, positive or negative.*

* For a proof of the binomial expansion for any exponent, see Fine's *College Algebra*, p. 553.

The binomial series is convergent for $|x| < 1$, and divergent for $|x| > 1$. For, we have

$$a_{n+1} = \frac{m(m-1)(m-2)\cdots(m-n)}{(n+1)!},$$

$$a_n = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!},$$

$$\frac{a_{n+1}}{a_n} = \frac{m-n}{n+1},$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{m-n}{n+1} = \lim_{n\to\infty} \frac{\frac{m}{n} - 1}{1 + \frac{1}{n}} = -1.$$

Hence, from Art. 168, the series converges for $-1 < x < 1$.

In expanding $(b+x)^m$ for fractional or negative values of m, we may write it in the form $b^m\left(1+\frac{x}{b}\right)^m$. The expansion will then proceed in powers of $\frac{x}{b}$, thus:

$$b^m\left(1+\frac{x}{b}\right)^m = b^m\left(1 + m\frac{x}{b} + \frac{m(m-1)}{2!}\left(\frac{x}{b}\right)^2 + \cdots\right).$$

This series converges when $\left|\frac{x}{b}\right| < 1$, that is, the interval of convergence for the expansion of $(b+x)^m$ is the interval between $-b$ and $+b$ (Fig. 38).

FIG. 38.

EXERCISES

Expand the following binomials to five terms and indicate the interval for which the expansion holds.

1. $(1+3x)^{-2}$.
2. $(1+2x)^{\frac{1}{2}}$.
3. $\sqrt[3]{1+x}$.
4. $(2-3x)^{-\frac{1}{2}}$.
5. $(2+x)^{\frac{3}{4}}$.
6. $\dfrac{1}{\sqrt{1-8x}}$.
7. $(5-4x^2)^{-\frac{2}{3}}$.
8. $(1-x)^{-\frac{7}{4}}$.
9. $(1+4x^2)^{-1}$.
10. $(3a+2x)^{\frac{5}{6}}$.

Extract the following roots to four places of decimals by the binomial expansion.

11. $\sqrt[3]{65}$.

Solution:

$$\sqrt[3]{65} = (4^3+1)^{\frac{1}{3}} = 4\left(1+\frac{1}{4^3}\right)^{\frac{1}{3}}$$
$$= 4\left(1+\frac{1}{3}\cdot\frac{1}{4^3}-\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{4^6}+\cdots\right)$$
$$= 4(1+0.00521-0.00003+\cdots)$$
$$= 4.0207+.$$

12. $\sqrt{5}$. **13.** $\sqrt{15}$. **14.** $\sqrt{1.02}$.

15. $\sqrt[3]{998}$. **16.** $\sqrt[6]{65}$. **17.** $\sqrt[3]{130}$.

18. $\sqrt[3]{731}$. **19.** $\sqrt[3]{258}$. **20.** $\sqrt[3]{0.515}$.

170. Exponential series. The power series

$$1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots$$

is called the **exponential series.** It is convergent for all values of x. For, we have

$$a_{n+1}=\frac{1}{n!},\quad a_n=\frac{1}{(n-1)!},\quad \frac{a_{n+1}}{a_n}=\frac{1}{n},$$

hence,
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=0.$$

It can be proved that

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
$$=e^x,$$

where
$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$$
$$=2.71828,$$

correct to five places of decimals.

This number e is the base of the natural system of logarithms.

Let
$$a^x=e^h,$$

then
$$x\log_e a=h.$$

Hence,

$$a^x=e^{x\log_e a}=1+x\log_e a+\frac{x^2(\log_e a)^2}{2!}+\frac{x^3(\log_e a)^3}{3!}+\cdots.$$

171. Logarithmic series. The power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \pm \cdots$$

is called the **logarithmic series.**

Since $$a_{n+1} = \pm\frac{1}{n+1},\ a_n = \mp\frac{1}{n},$$

we have $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = -1,$$

and the series is convergent for $|x| < 1$.

FIG. 39.

It will be shown in the calculus * that this series converges to $\log_e(1+x)$ for any value of x for which the series is convergent. The series can then be used to find logarithms of numbers to the base e. Thus,

$$\log_e(\tfrac{3}{2}) = \log_e(1+\tfrac{1}{2})$$
$$= \frac{1}{2} - \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^4}{4} + \cdots.$$

The logarithmic series can be used to calculate logarithms of positive numbers less than 2. However, it converges so slowly that it is not well adapted to numerical computation.

To derive a more convenient series for the calculation of natural logarithms, we proceed as follows: †

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Hence, $$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

By subtraction, $$\log_e(1+x) - \log_e(1-x) = \log_e\frac{1+x}{1-x}$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

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* Townsend and Goodenough's *First Course in Calculus*, p. 326.

† For a more detailed discussion, see Osgood's *Introduction to Infinite Series*, pp. 23 and 44.

Let $$\frac{1+x}{1-x}=\frac{m+1}{m},$$

whence $$x=\frac{1}{2m+1}.$$

We have then

$$\log_e\frac{m+1}{m}=2\left(\frac{1}{2m+1}+\frac{1}{3(2m+1)^3}+\frac{1}{5(2m+1)^5}+\cdots\right),$$

or $$\log_e(m+1)=\log_e m+2\left(\frac{1}{2m+1}+\frac{1}{3(2m+1)^3}+\frac{1}{5(2m+1)^5}+\cdots\right).$$

If $m=1$, we have

$$\log_e 2=0+2\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\cdot 3^5}+\cdots\right)=0.6931+\cdots.$$

Letting $m=2$,

$$\log_e 3=\log_e 2+2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\cdots\right)$$

$$=0.6931+0.4055+\cdots=1.0986+\cdots.$$

In this way the logarithm of any number to the base e may be computed.

From Art. 120, if a is any positive number, we have

$$\log_{10} a=\frac{\log_e a}{\log_e 10}=\frac{1}{\log_e 10}\cdot\log_e a.$$

Hence, if we have computed the logarithm of a number to the base e, we can find its logarithm to the base 10 by multiplying by $\frac{1}{\log_e 10}$. To five significant figures, $\frac{1}{\log_e 10}=0.43429$.

In computing a table of logarithms we need compute the logarithms of prime numbers only. The logarithms of composite numbers may then be found by means of the theorems on logarithms.

EXERCISES

1. By computing the logarithms of 2, 3, 5, 7, construct a table of logarithms to the base 10 for the numbers 1 to 10.

2. Using the series for $\log_e(1+x)$, compute $\log_e\frac{4}{3}$ to three places of decimals.

3. Find $\log_e 16$ and $\log_e 17$ to four significant figures.

ANSWERS

[The answers to some exercises are intentionally omitted.]

Article 6. Page 11

1. 5. **2.** 9. **3.** 6. **4.** 3, 2, 3.

5. 5, 4. **6.** 12, 5. **7.** a. **8.** 1.

9. $\frac{x^5}{y^5}$. **10.** m^4n^7. **11.** $\frac{b^8}{16\,a^2x^6}$.

12. $8\,a^3 + 12\,a^2b^2 + 6\,ab^4 + b^6$. **13.** a^{3n}. **14.** a^{3n}.

15. a^n. **16.** a^{2n+2}. **17.** a^{2n-2}. **18.** $\left(\frac{b}{a}\right)^n$.

19. $\frac{(-3)^n x^{2n}}{4^n y^n}$. **20.** $\frac{9^n x^{4n}}{16^n y^{2n}}$. **21.** $2\frac{m}{n} + \frac{n}{m}$.

22. $x^2y^2 + xy + 1$. **23.** $a^{2n} - b^{2n}$. **24.** $x^{2n-2} - y^{2n-2}$.

25. $p^{2n-2} - p^{2n-5} + p^{2n-1} - p^{2n-4}$. **26.** $a^{6n-9} \cdot x^{12n-3} \cdot y^{9n-3}$.

27. $x^{3n} \cdot P^{6n} \cdot y^{n^2}$. **28.** x^6. **29.** 1.

30. $\frac{7\,a^{4-m} \cdot x^{n-3}}{15}$. **31.** a^{5n}. **32.** $x^{n^2} + y^{n^2}$.

33. $4\,a^{2n} + 2\,a^nb^n + b^{2n}$. **34.** $x^{2n} - x^n + 1$.

35. $(a^2 + ab + b^2)(x + y)^2$. **36.** $27\,p^{3n} - 8\,q^{3n}$.

Article 12. Pages 15–17

1. $\frac{5\,x^3}{a^3}$. **2.** $5\,a^3x^3$. **3.** $\frac{1}{8}$. **4.** $\frac{1}{8}$.

5. 3. **6.** $\frac{4}{9}$. **7.** $\frac{5}{4}$. **8.** $\frac{25}{16}$.

9. $\frac{5}{2}$. **10.** 0.7. **11.** $1\frac{3}{7}$. **12.** $\frac{3^4}{2 \cdot 5^2}$.

13. $\frac{x}{a}$. **14.** $\frac{10\,b^7}{a^7c^3}$. **15.** $a^{\frac{1}{4}}$. **16.** $a^{\frac{2}{3}}b^4c$.

17. $2\,xy^3$. **18.** $\frac{x^2}{y}$. **19.** $\frac{a^{0.06}}{x^{0.14}}$. **20.** $\frac{x^{\frac{14}{3}}y^{\frac{1}{2}}}{a^{\frac{5}{3}}}$.

21. $-x^{14}y^7z^{-21}$. **22.** $p^{-0.14}$. **23.** $x - y$. **24.** $1 - b$.

25. $a^{\frac{4}{3}} - b^{\frac{4}{3}}$. **26.** $x^{\frac{7}{3}}y^{\frac{5}{3}} + x^{\frac{5}{3}}y^{\frac{1}{3}}$. **27.** $x^{-1} - y^{-1}$. **28.** $1 + x^{\frac{3}{2}}$.

29. $a - \frac{1}{a}$. 30. $a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$. 31. $a + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} + b$.

32. $x^{-4} + x^{-2} + 1$. 33. $p^{-3} - 3p^{-2} + 3p^{-1} - 1$.

34. $2 + x^{\frac{1}{4}} + x^{-\frac{1}{4}}$. 35. $2x^{\frac{5}{2}}y^{-\frac{1}{2}} - 5y^{\frac{1}{2}}x^{-\frac{1}{2}}$.

36. $x^{-1} + 8$. 37. $m^{-2} - m^{-1} + 1$.

38. $x^{-3} - 2x^{-2}y^{-1} + 4x^{-1}y^{-2} - 8y^{-3}$.

39. $a^{-8} + a^{-7} - a^{-5} - a^{-4} - a^{-3} + a^{-1} + 1$.

40. $x + x^{\frac{1}{2}}$. 41. $a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2a^{\frac{1}{3}} - 3b^{\frac{1}{2}}$. 42. $x^{-\frac{1}{3}} + a^{\frac{1}{4}} + a^{\frac{2}{3}}x^{\frac{1}{3}}$.

43. $\sqrt{x}\sqrt[3]{y^2}$. 44. $\frac{\sqrt[4]{b}}{\sqrt{a}}$. 45. $\sqrt[6]{(a+b)^5}$.

46. $\frac{1}{\sqrt[6]{(a-b)^5}}$. 47. $\frac{\sqrt[m]{x^n}}{y}$. 48. $\frac{b^2}{\sqrt[3]{a^{10}}}$.

49. 7. 50. $y\sqrt{x^9z}$. 51. $\frac{1}{\sqrt{x}\sqrt[4]{az^3}}$.

52. $x^{\frac{5}{2}}yz^{\frac{3}{2}}$. 53. $a^{-4}x^{-\frac{4}{3}}$. 54. $a^{\frac{7}{3}}x^{\frac{1}{2}}$.

55. $x^{\frac{4}{3}}z^2$. 56. $\frac{a^{\frac{2}{3}}b^{\frac{4}{3}}}{x}$. 57. $2x^{\frac{3}{4}}$.

58. $2^{\frac{1}{4}}$. 59. $a^n x^{\frac{2}{n}} y$. 60. $\frac{x^{1-\frac{1}{n}} \cdot y^{\frac{m}{n}}}{z^2}$.

61. $a^{\frac{2}{n^2-1}}$. 62. $a^{\frac{p^2-q^2+2}{2p}}$. 63. $x^{\frac{7}{3}}$.

64. $a^{9n} \cdot b$. 65. $x^{2\frac{5}{8}}$. 67. 170.4.

68. 10^{-6}, $3 \cdot 10^{-9}$, $15 \cdot 10^{-7}$. 69. $3\frac{3}{4}$, $-\frac{1}{2}$, 0, $3\frac{3}{4}$. 71. $666 \cdot 10^{-10}$.

72. 10^{-6}. 73. 0.6867, 0.5896, 0.4861, 0.3934 microns.

74. 1 preceded by twenty-four zeros after the decimal point.

Article 14. Page 19

1. $5a + 12b$. 2. $a - b + c + d - e$. 3. $6x + 2$.

4. $x + 2y$. 5. 0.

6. $4c + 3b - (5c^2 - 4a - 4d^3)$. 7. $a - d - (c - b)$.

8. 20. 9. $6a + 5b - (a + b) - (4a + 3b)$.

10. $(x^2 + y^2)^2$.

Article 15. Pages 20–21

1. $\frac{z}{z-y}$. 2. $\frac{x+1}{x}$. 3. $\frac{2ab}{a^2+b^2}$. 4. $\frac{2}{2m-1}$.

5. $\frac{1-a}{a}$. 6. $\frac{x-1}{x+1}$. 7. $\frac{x^2+x+1}{x}$. 8. $\frac{xz-y^2}{xz+y^2}$.

9. $\frac{m+n}{m-n}$. **10.** $\frac{1}{x+1}$. **11.** $x+y$. **12.** $x+1$.

13. $x+y$. **14.** $\frac{x^2y}{x+y}$. **15.** $\frac{x-2}{x+2}$. **16.** 2.

17. x^2-x+1. **18.** $\frac{2ax}{a^2+x^2}$. **19.** $x-1$. **20.** n.

21. $\frac{1}{x(x+h)}$. **22.** $-\frac{1}{(x-1)(x+h-1)}$. **23.** $\frac{2x+h}{x^2(x+h)^2}$.

24. $-\frac{1}{rs(r+s)}$. **25.** $\frac{E}{3r+Rt}$. **26.** $\frac{6p+14R}{R}$.

27. $\frac{x}{a}$. **28.** a^2b. **29.** x. **30.** $\frac{a+b}{ab}$.

31. $a^4b^2xy^3\left(\frac{1-2y^3x^2}{y^3-2x^3}\right)$. **32.** $a+b$.

33. $(a^2-x^2)^{\frac{1}{2}}$. **34.** $\frac{a^2-x^2}{x}$.

Article 17. Pages 24–25

2. $\sqrt{99}$. **3.** $\sqrt[3]{40}$. **4.** $\sqrt[4]{112}$. **5.** $\sqrt[3]{104}$. **6.** $\sqrt[3]{2058}$.

7. $\sqrt[3]{192}$. **10.** $\frac{1}{5}\sqrt[3]{25}$. **11.** $\frac{1}{5}\sqrt[4]{250}$. **12.** $\frac{1}{7}\sqrt{28}$. **13.** $\sqrt[5]{81}$.

14. $3\sqrt[4]{14}$. **15.** $\sqrt{42}$. **18.** $2\sqrt[3]{4}$. **19.** $3\sqrt{2}$. **20.** $5\sqrt{5}$.

21. $2\sqrt[4]{3}$. **22.** $3\sqrt[3]{3}$. **23.** $20\sqrt{2}$. **24.** $3\sqrt[6]{3}$. **27.** $\sqrt{20}$.

28. $\sqrt[3]{9}$. **29.** $\sqrt{13}$. **30.** $\sqrt{7}$. **31.** $\sqrt[4]{8}$. **34.** $\frac{\sqrt{5}}{5}$.

35. $2\sqrt[3]{4}$. **36.** $\frac{1+\sqrt{2}}{-2}$. **37.** $15-5\sqrt{8}$.

38. $3\sqrt{5}-3\sqrt{3}$. **39.** $\frac{5}{9}(\sqrt{20}+\sqrt{2})$. **40.** $-\frac{1}{5}(\sqrt{6}+\sqrt{21})$.

41.
$$\begin{cases} \frac{1}{1+\sqrt{2}} = 0.4142. \\ \frac{1}{\sqrt{5}} = 0.4472. \\ \frac{1}{1-\sqrt{3}} = -1.3660. \\ \frac{4}{\sqrt{2}-\sqrt{3}} = 12.585 \end{cases}$$

Article 18. Pages 25–26

2. $\frac{1}{5}\sqrt[3]{300}$. **3.** $\frac{1}{7}\sqrt{35}$. **4.** $\frac{1}{11}\sqrt[3]{363}$. **5.** $9\sqrt{15}$. **6.** $\sqrt{2}$.

7. $\frac{26-7\sqrt{3}}{23}$. **8.** $\frac{5}{13}(5-2\sqrt{3})$. **9.** $\frac{\sqrt{6}+\sqrt{15}}{3}$. **10.** -5.

11. $5+2\sqrt{3}$. **12.** $\frac{28-\sqrt{3}}{33}$. **13.** $\sqrt{a}+1+\frac{\sqrt{a}}{a}$. **14.** $x^{\frac{2}{3}}+2$.

Article 19. Page 26

2. $3\sqrt{3}$. **3.** $\sqrt{6}$. **4.** $x\sqrt{3x}$. **5.** $9\sqrt{3}$. **6.** $\sqrt{2}$. **7.** $(b+x-ay)\sqrt{a}$.
8. $\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2}$. **9.** $\frac{11}{5}\sqrt{5}$. **10.** $9\sqrt[3]{2x}$. **11.** $7\sqrt{a}-2\sqrt[3]{a}$.
12. $16(3)^{\frac{1}{2}}$. **13.** $15(3)^{\frac{1}{2}}$. **14.** $3(5)^{\frac{1}{2}}$. **15.** $5^{\frac{1}{3}}$.

Article 20. Page 27

2. $\sqrt[6]{675}$. **3.** $a\sqrt[6]{72a}$. **4.** 100. **5.** $24\sqrt[3]{6}$. **6.** $\sqrt[5]{77}$. **7.** $12\sqrt[12]{a^4b^3}$.
8. 6. **9.** $12+7\sqrt{6}$. **10.** $2\sqrt{3}-\sqrt{10}$. **11.** $m+3\sqrt[3]{m^2n}+3\sqrt[3]{mn^2}+n$.
12. 5. **13.** $3-5a+2\sqrt{a(1+a)}$. **14.** x^4+1. **15.** $\sqrt[12]{x^4y^6z^3}$.
16. $\sqrt[2n]{a^7}$. **17.** $5\sqrt[3]{4}$. **18.** $\sqrt[3]{3}$. **19.** $2\sqrt[12]{108}$. **20.** $2\sqrt[6]{3}$. **21.** $\sqrt[4]{\frac{x}{y}}$.
22. $\frac{5}{3}\sqrt{5}$. **23.** $\frac{\sqrt{ab}}{a}$. **24.** $\frac{1}{3}\sqrt[4]{1512x}$. **25.** $6^{\frac{1}{3}}$. **26.** $7^{\frac{1}{6}}$.
27. $7^{-\frac{1}{6}}=\frac{1}{7}\sqrt[6]{7^5}$. **28.** $\sqrt[8]{\frac{a}{5}}=\frac{1}{5}\sqrt[8]{5^7a}$. **29.** $\frac{1}{2}\sqrt[6]{18a}$. **30.** $\sqrt[10]{96ax^2}$.

Article 21. Pages 27–30

1. 2288 feet per sec. **2.** 25.30. **3.** 500.7. **4.** 10.71. **5.** 0.743.
6. 0.0205. **7.** 1,9 feet. **8.** 9.39. **9.** 26.95. **10.** 29.83. **11.** 154.
13. 1077. **14.** 30. **15.** 50.9.

Article 22. Page 30

2. 5. **3.** 1. **4.** $-2-2i\sqrt{3}$. **5.** x^2+a^2. **6.** $-i$. **7.** 7.
8. 625. **9.** i. **10.** 0. **11.** 14. **12.** $2a(a^2-3b^2)$. **13.** 1054.
14. 0. **15.** $(x-a)^2+b^2$.

Article 25. Pages 32–33

1. $A=\pi r^2$. **2.** $V=f(x)$. **3.** $y=\sqrt{100-x^2}$. **4.** $P=\frac{C}{V}$, $C=f(t)$.

5. a^2+a+1, 1, 3, 1, 91, a^2+3a+3. **6.** 3, -1, 0, $1-\sqrt{2}$, $\frac{a+b+1}{a+b-1}$.

7. -1, $\frac{1+x^2}{1-x^2}$, $\frac{x+1}{x-1}$. **9.** x^4+x^2+1, x^2+3x+3,

$x^4+2x^3+4x^2+3x+3$. **11.** $\frac{1+2t}{3+4t}$. **12.** $\frac{7t+10}{15t+22}$.

Article 26. Page 35

5. $(-12, -3)$.

6. $(-1, 4)$, $(2, 4)$, $(5, 4)$, $(5, 1)$, $(5, -2)$, $(2, -2)$, $(-1, -2)$, $(-1, 1)$.

8. (*a*) $(-1, 5)$, $(3, 5)$, $(3, 1)$, $(-1, 1)$.
(*b*) $(1, 5.83)$, $(3.83, 3)$, $(1, 0.17)$, $(-2.83, 3)$.

Article 29. Page 40

2. $\frac{5}{3}$. **3.** $-\frac{3}{4}$. **4.** 1, 5. **5.** 0, -4.
6. 4, $-\frac{1}{3}$. **7.** 1, 5. **8.** 1, -1, -3. **9.** 0, 5, -5.
10. 0 and 1; -1 and 0; -6 and -5.

Article 36. Pages 47–48

1. (*a*) $2x^2 = 5x$ with the additional root 0. (*b*) $2x(x-3) = 5(x-3)$ with the additional root 3. (*c*) $2x(x^2-4) = 5(x^2-4)$ with the additional roots 2 and -2. **2.** $(x^2+2)(x-3) = 3x(x-3)$. This equation is redundant with respect to a root 3. **3.** 0. **4.** (*a*) $x^2-6=3$ or $x^2=9$; (*b*) 2. **5.** $8x^2-24x+16=0$. **6.** a. **7.** a. **8.** $7x-1=0$.
9. $7x=12$. **10.** $3x+8=0$. **11.** $x-5=0$. **12.** $x-25=0$.
13. $x-64=0$. **14.** $x^2-5x-14=0$. **15.** $(x-1)(7x^2-16x-15)=0$.
16. $7x-75=0$. **17.** $x=1$.

Article 38. Page 51

2. $x=1\frac{1}{2}$, $y=\frac{7}{4}$. **3.** $x=16$, $y=15$. **4.** $x=1$, $y=2$. **5.** $x=7$, $y=2$. **6.** $x=7$, $y=11$. **7.** $x=6$, $y=12$.

Article 39. Pages 52–53

2. $x=1$, $y=1$. **3.** $x=3$, $y=4$. **4.** $x=1$, $y=3$. **5.** $x=1$, $x=-1$. **6.** $x=2$, $y=3$. **7.** $x=4$, $y=2$. **8.** $x=1$, $y=2$.
9. $x=18$, $y=10$. **10.** An infinitely large number of solutions. **11.** No solution. **12.** $x=\frac{1}{3}$, $y=\frac{3}{4}$. **13.** $x=0$, $y=\frac{3}{7}$. **14.** $x=5$, $y=2$.

Article 40. Page 54

2. 154. **3.** 41. **4.** 39. **5.** 15. **6.** $axv+byu$. **7.** $x^2-30x+40$.
8. 0.

Article 41. Page 56

2. $x=1$, $y=2$, $z=3$. **3.** $x=10$, $y=8$, $z=6$. **4.** $x=10$, $y=2$, $z=3$. **5.** $x=2$, $y=-1$, $z=3$. **6.** $x=2$, $y=2$, $z=2$. **7.** $x=-4$, $y=2$, $z=-5$. **8.** No solution. **9.** $x=7$, $y=4$, $z=3$.

Miscellaneous Exercises and Problems. Pages 56–58

3. 0. **4.** $2y-2x$. **5.** 6. **6.** $x=\frac{5}{3}$. **7.** $x=2$, $y=5$.
8. 30 gallons of 25 per cent alcohol and 20 gallons of 30 per cent alcohol.
9. 3 oz. and 5 oz. **10.** 18° and 72°. **11.** 512,000 sq. ft. **12.** 27 and 72.
13. 428,571. **14.** 5 mo. **15.** 6900. **17.** 5 min. $29\frac{1}{5}$ sec.; 6 min. 45 sec. **19.** $c=0.036$, $b=999.5$. **20.** \$20,000, \$15,000. **21.** Analytic geometry 85, algebra 95, trigonometry 90.

22. 2, 3, and 6 hr. **23.** 12, 9, 3, 21. **24.**
2, 7, 12, 17, 22.
0, 6, 12, 18, 24.
4, 8, 12, 16, 20.
6, 9, 12, 15, 18.
8, 10, 12, 14, 16.
10, 11, 12, 13, 14.

25. First 8, second 15. **26.** 1020 bonds.

27. Going 9 miles uphill, $13\frac{1}{2}$ miles level, $4\frac{1}{2}$ miles downhill.

Article 44. Pages 62–63

1. 7, 2. **2.** $-1, -7$. **3.** $5, -4$. **4.** $-\frac{1}{2}$.

5. $-\frac{8}{4}, \frac{1}{5}$. **6.** $5, -\frac{2}{3}$. **7.** $\frac{2}{2}, -4$. **8.** $1, -\frac{3}{2}$.

9. $-2, \frac{4}{7}$. **10.** $-6, \frac{1}{6}$. **11.** $7, -2$. **12.** $-2 \pm \frac{3}{2}i$.

13. $3 \pm i$. **14.** $\frac{7}{2}, \frac{7}{2}$. **15.** $\frac{14}{3}, -\frac{5}{3}$. **16.** $a, -a-1$.

17. 5, 0. **18.** $3 \pm \sqrt{5}$. **19.** $-\frac{1}{3} \pm \frac{i}{3}\sqrt{2}$. **20.** $\frac{m}{1+e}, \frac{m}{1-e}$.

21. 9. **22.** 10. **23.** 5. **24.** 2.

25. 5. **26.** 50. **27.** 16, 23. **28.** $\frac{1}{6}, \frac{3}{2}$.

29. 9, 33. **30.** No solution. **31.** 1.6– or 0.75. **32.** 3.1 or -2.2.

33. 16 or 6.2. **34.** 5.4 or -0.51.

Article 45. Pages 64–65

2. $\pm 2, \pm 3$. **3.** $\pm \frac{1}{2}\sqrt{7}, \pm \frac{i}{2}$. **4.** $\pm 2, \pm 2i$.

5. $1, -6, \frac{-5 \pm i\sqrt{3}}{2}$. **6.** $1, 1, -3 \pm 2\sqrt{2}$.

7. 0. **8.** 10. **9.** $\frac{3}{2}, -\frac{1}{2}$. **10.** 25, 9.

11. $\pm \frac{7}{6}$. **12.** $-1, 3$. **13.** $-2a, a, \frac{a}{2}(-1 \pm i\sqrt{3}), a(1 \pm i\sqrt{3})$.

14. $\frac{-1 \pm \sqrt{13}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$. **15.** $2 \pm 2i, 2 \pm \sqrt{5}$.

16. 1. **17.** $\frac{1}{4}$, 1. **18.** 1. **19.** $(-1 \pm \frac{1}{3}\sqrt{21})^3$.

20. $2, -1 \pm i\sqrt{3}$. **21.** $\sqrt[n]{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$.

22. $2, -\frac{3}{2}$. **23.** 0, 49. **24.** 1.

25. $0, -5, \frac{-5+\sqrt{-15}}{2}$.

Article 46. Pages 65–66

2. $x^2 - x - 6 = 0$. **3.** $x^2 - 7x = 0$.

4. $x^2 + (5-\sqrt{5})x - 5\sqrt{5} = 0$. **5.** $x^2 - 5 = 0$.

6. $x^2 - x \cdot 2\sqrt{5} + 4 = 0$. **7.** $x^2 + 1 = 0$.

8. $x^2 - 8x + 25 = 0$.
9. $x^2 - 4x - 71 = 0$.
10. $x^2 + x + 1 = 0$.
11. $x^2 - \frac{5}{2}x + 1 = 0$.
12. $x^2 - \frac{(a^2+1)x}{a} + 1 = 0$.
13. $x^2 - \left(\frac{m^2+n^2}{mn}\right)x + 1 = 0$.

Article 48. Page 68

1. $k = \pm\sqrt{5}$.
2. $k = 8$.
3. $k = 3$, or 1.
4. $m = -\frac{1}{2}$, $k = -8$.
5. $k = -1$, $m = \frac{1}{4}$.
6. $k = 0$, or $-\frac{1}{2}$, $m = -\frac{1}{2}$.
7. $k = -\frac{1}{3}$.
8. $k = \pm 2$.
9. $k = 1$.
10. $k = 3$.
11. $k = 4$.
12. $k = 3$, or -1.

Article 50. Pages 69–70

1. Real and unequal.
2. Imaginary.
3. Real and equal.
4. Real and unequal.
5. Real and equal.
7. $k = \pm 4$.
8. $k = 9$.
9. $k = 9$.
10. $k = \pm 5$.
11. $k = -2$, or $-\frac{2}{3}$.
12. $k = \frac{3}{2}$.
13. $k = 1$.
14. $k = \pm r\sqrt{1+m^2}$.
15. $k = \pm\sqrt{a^2m^2+b^2}$.
16. $-\frac{1}{7}$, $-\frac{3}{7}$.
17. $-\frac{3}{2}$, $-\frac{11}{18}$.
18. $-\frac{2}{m}$, $-\frac{6}{m}$.
19. $-\frac{7}{8}$, $-\frac{31}{4}$.
20. $\frac{2m}{1-e^2}$, $\frac{m^2}{1-e^2}$.
22. $k = 3$.
23. $k = 20$.
24. $k = -4$.
25. $k = 1$.
26. $k = \pm 13$.

Problems. Pages 71–74

2. $5\frac{1}{2}$ per cent.
3. 50×75.
4. 33.89, 81.89.
5. 0.8^+ sec. or 2.3^- sec.
6. 3.1^+ sec.
7. 1.5^+ sec.
8. 38.8^+ ft.
9. One half the whole time.
10. 4.9 ft. Ball begins to fall before the end of the second second.
11. 3.7^- sec.
12. 80.5 ft. per sec.
13. 144.9 ft.
14. 2 in.
15. 5.828^+.
16. $12 + 7\sqrt{3} = 24.125$.
18. 0.625.
21. $\frac{11}{16}$.
22. 25.
23. 14×7.
24. 1 ft. 4 in.
25. 1 ft. 2 in.
26. 8 ft. per sec.
27. 33.47.
28. 238 ft. if time is measured to nearest tenth of a second.

29. $198°-$. 30. 5. 31. $R=\frac{1\pm\sqrt{1-4g^2X^2}}{2g}$.

32. $n-n^1=\frac{a\pm\sqrt{a^2-4T^2b}}{2Tb}$. 33. $x=\frac{rPX\pm\sqrt{P^2X^2+4R^2W(P-W)}}{2RW}$.

34. 1 part of corn to 4 parts rye.

Article 53. Case I, Page 77

2. $(\frac{5}{4}\sqrt{7}, \frac{9}{4})$, $(\frac{5}{4}\sqrt{7}, -\frac{9}{4})$, $(-\frac{5}{4}\sqrt{7}, \frac{9}{4})$, $(-\frac{5}{4}\sqrt{7}, -\frac{9}{4})$.
3. $(0, 3)$, $(0, -3)$.
4. $\left(\frac{5i}{4}\sqrt{5}, \frac{3}{4}\sqrt{21}\right)$, $\left(\frac{5i}{5}\sqrt{5}, -\frac{3}{4}\sqrt{21}\right)$, $\left(-\frac{5i}{4}\sqrt{5}, \frac{3}{4}\sqrt{21}\right)$, $\left(-\frac{5i}{4}\sqrt{5}, -\frac{3}{4}\sqrt{21}\right)$.
5. $(5, 0)$, $(-5, 0)$.
6. $(\frac{15}{34}\sqrt{34}, \frac{15}{34}\sqrt{34})$, $(\frac{15}{34}\sqrt{34}, -\frac{15}{34}\sqrt{34})$, $(-\frac{15}{34}\sqrt{34}, \frac{15}{34}\sqrt{34})$, $(-\frac{15}{34}\sqrt{34}, -\frac{15}{34}\sqrt{34})$.
7. $(\frac{6}{13}\sqrt{65}, \frac{2}{13}\sqrt{91})$, $(\frac{6}{13}\sqrt{65}, -\frac{2}{13}\sqrt{91})$, $(-\frac{6}{13}\sqrt{65}, \frac{2}{13}\sqrt{91})$, $(-\frac{6}{13}\sqrt{65}, -\frac{2}{13}\sqrt{91})$.
8. $(0.32, 3.1)$, $(-0.32, -3.1)$, $(0.32, -3.1)$, $(-0.32, 3.1)$.
9. $(1.1, 2.3)$, $(-1.1, 2.3)$, $(1.1, -2.3)$, $(-1.1, -2.3)$.

Case II, Pages 78–79

2. $(1, -2)$.
3. $\left(\frac{-1-8i\sqrt{6}}{5}, \frac{-18+6i\sqrt{6}}{5}\right)$, $\left(\frac{-1+8i\sqrt{6}}{5}, \frac{-18-6i\sqrt{6}}{5}\right)$.
4. $(3, 7)$, $(7, 3)$.
5. $(4, 9)$, $(9, 4)$.
6. $(8, 5)$, $(-5, -8)$.
7. $(1, 6)$, $(-\frac{1}{7}, -2)$.
8. $(4, 10)$, $(100, -110)$.
9. $(1, 2)$.
10. $(1, 1)$, $(-\frac{2}{3}, -\frac{7}{3})$.
11. $(2, 3)$, $(-\frac{1}{3}, \frac{2}{3})$.
12. $(0, 0)$, $(4, 5)$.
13. $(0, 0)$, $(-1, -2.4)$.
14. $(1.4, -0.17)$, $(0.034, -7.1)$.
15. $(1.1, 2.2)$, $(0.03, -0.71)$.
17. $r>5\sqrt{2}$, $r=5\sqrt{2}$, $r<5\sqrt{2}$.
18. $c<25$, $c=25$, $c>25$.
19. Two points for any real value of a.
20. Two points for any real value of b.
21. $b=\pm r\sqrt{1+m^2}$.
22. $k=\pm\sqrt{a^2m^2+b^2}$.

Case III, Pages 80–81

2. $(4, 1)$, $(-4, -1)$, $(14, -4)$, $(-14, 4)$. 3. $(-\sqrt{6}, \sqrt{6})$, $(\sqrt{6}, -\sqrt{6})$.
4. $(3, 5)$, $(-3, -5)$, $(\frac{5}{3}, \frac{13}{3})$, $(-\frac{5}{3}, -\frac{13}{3})$.
5. $(4, 7)$, $(-4, -7)$, $(7, 4)$, $(-7, -4)$.

6. $(3, 5), (-3, -5), (8, -5), (-8, 5)$.
7. $(\frac{5}{7}\sqrt{21}, \frac{1}{7}\sqrt{21}), (-\frac{5}{7}\sqrt{21}, -\frac{1}{7}\sqrt{21})$.
8. $(6, 3), (-6, -3), (6, -3), (-6, 3)$.
9. $(4, 5), (-4, -5), (3\sqrt{3}, \sqrt{3}), (-3\sqrt{3}, -\sqrt{3})$.
10. $(2, 4), (-2, -4), \left(\frac{18i}{101}\sqrt{101}, \frac{-16i}{101}\sqrt{101}\right), \left(\frac{-18i}{101}\sqrt{101}, \frac{16i}{101}\sqrt{101}\right)$.
11. $(\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}), (-\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5})$.
12. $(2.7, 0.85), (2.7, -0.85), (-2.7, 0.85), (-2.7, -0.85)$.
13. $(10, 1.4), (-10, -1.4), (11, 17), (-11, -17)$.

Case IV. Page 82

2. $(5, 0), (0, 5), (1, -4), (-4, 1)$.
3. $(3, 2), (2, 3), \left(\frac{-1+i\sqrt{23}}{6}, \frac{-1-i\sqrt{23}}{6}\right), \left(\frac{-1-i\sqrt{23}}{6}, \frac{-1+i\sqrt{23}}{6}\right)$.
4. $(3, 1), (1, 3), \left(\frac{-19+\sqrt{309}}{8}, \frac{-19-\sqrt{309}}{8}\right), \left(\frac{-19-\sqrt{309}}{8}, \frac{-19+\sqrt{309}}{8}\right)$.
5. $(2, 3), (3, 2), (-5+i\sqrt{11}, -5-i\sqrt{11}), (-5-i\sqrt{11}, -5+i\sqrt{11})$.
6. $(3, 2), (2, 3), (-3, -2), (-2, -3)$.

Exercises and Problems. Pages 82-85

1. $(3, 1), (-\frac{5}{17}, -\frac{15}{17})$.
2. $(2, \frac{1}{2}), (-\frac{2}{3}, -\frac{1}{6})$.
3. $(\frac{1}{13}, \frac{1}{8}), (-\frac{1}{8}, -\frac{1}{13})$.
4. $(1, 2), (-\frac{5}{32}, -\frac{21}{8})$.
5. $(2, 7), (18, \frac{5}{3})$.
6. $(3, 5), (-3, -5), (\frac{5}{3}, \frac{13}{8}), (-\frac{5}{3}, -\frac{13}{8})$.
7. $(6, 2), (3, 5)$.
8. $(2, 3), (-\frac{52}{289}, -\frac{33}{289})$.
9. $(2, 11), (\frac{7}{4}, 12)$.
10. $(5, 4), (\frac{71}{4}, -\frac{69}{4})$.
11. $(9, 5), (-\frac{43}{7}, -\frac{81}{5})$.
12. $(1, 2), (\frac{1}{2}, \frac{5}{2}), (-1+i\sqrt{3}, -3-i\sqrt{3}), (-1-i\sqrt{3}, -3+i\sqrt{3})$.
13. $(6, 2), \left(-\frac{9}{2}, -\frac{3}{2}\right), \left(\frac{5+\sqrt{265}}{4}, \frac{5+\sqrt{265}}{4}\right), \left(\frac{5-\sqrt{265}}{4}, \frac{5-\sqrt{265}}{4}\right)$.
14. $(2, 5), (\frac{7}{6}, \frac{11}{2})$.
15. $(7, 1), (-49, -13), (i\sqrt{11}, -i\sqrt{11}), (-i\sqrt{11}, i\sqrt{11})$.
16. $(4, 2), (-2, -1), \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(\frac{1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}\right)$.
17. $(-4, -6), (5, 3), \left(\frac{-9+\sqrt{141}}{2}, \frac{-3+\sqrt{141}}{2}\right), \left(\frac{-9-\sqrt{141}}{2}, \frac{-3-\sqrt{141}}{2}\right)$.

18. (3, 1), (1, 3). **19.** (3, 2), (− 2, − 3). **20.** (4, 1), (− 1, − 4).
21. (0, 1), (2, 5), $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$.
22. (9, 4), (4, 9), (6, 6), (− 6, − 6). **23.** (4, 1). **24.** (5, 9), (− 5, 9).
25. $(2a, a)$, $(-2a, -a)$. **26.** $(0, a)$, $(a, 0)$.
27. (2, 3, 4), (2, − 3, 4), (2, − 3, − 4), (2, 3, − 4), (− 2, 3, 4), (− 2, − 3, 4), (− 2, 3, − 4), (− 2, − 3, − 4).
28. $(2, 1, 1)$, $(\frac{61}{3}, -10, -\frac{74}{3})$. **29.** (1, 3, 5), (− 1, − 3, − 5).
30. (8, 3, 6, 4),* (8, 3, 4, 6), (3, 8, 4, 6), (3, 8, 6, 4).
31. (1, 2, 3, 5), (2, 1, 5, 3), $(\frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{3}{2})$, $(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, \frac{7}{2})$, (3, 5, 1, 2), (5, 3, 2, 1), $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2})$, $(\frac{3}{2}, \frac{7}{2}, \frac{5}{2}, \frac{1}{2})$.
32. (1, 1, 2, 2, 3),† (2, 1, 1, 3, 2). **33.** $b^2 + c^2 + 2a^2$.
34. $a^4 - b^4 = 4c^4$. **35.** $a^4 + b^4 = c^2(a^2 + b^2)$. **36.** $a^4 + b^4 - c^4$.
37. 628.5, 653.3. **38.** 300 and 400. **39.** 12, 6, and 4.
40. \$ 550, \$ 450 ; 7 %. **41.** 4.4 cents, 7.5 cents.
42. 0.1556. **43.** 10 and 12.
44. 24 lots, \$ 200 per lot. **45.** Radius, $2\sqrt{5}$, altitude, $8\sqrt{5}$.
46. 467.8. **47.** 9 × 13 × 21 ft.
48. 5.4 × 8.15 inches. **49.** 100 miles, 25 miles per hour.
50. $x^2 - 10x + 9 = 0$. **51.** 12 ft. 11 in., 16 ft. 1 in.
52. 7 and 5. **53.** 36 miles per hour.

Article 57. Page 89

1. $x > 5$. **2.** $x > \frac{8}{2}$. **3.** $x < 9$.
4. $-2 < x < 2$. **5.** $x > 4$ and $x < -\frac{1}{3}$. **6.** x positive but $\neq 1$.
7. $x < \frac{7}{3}$. **8.** $x > 5$ and $x < -2$. **9.** $-8 < x < \frac{8}{2}$.
10. $0 < x < a$, if $a > 0$; $a < x < 0$, if $a < 0$.
11. $x > -\frac{b}{a}$, if a is positive; $x < -\frac{b}{a}$, if a is negative.
12. $x^2 - 4x + 3 > 0$, if $x > 3$ or < 1.
$= 0$, if $x = 1$ or 3.
< 0, if $1 < x < 3$.
13. $ax^2 + bx + c > 0$, if a is positive and $x \neq \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, nor between these values.

$ax^2 + bx + c = 0$, if $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

$ax^2 + bx + c < 0$, if a is a positive and x lies between $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

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* (8, 3, 6, 4) means $w = 8$, $y = 3$, $z = 6$, $u = 4$.
† (8, 1, 2, 2, 3) means $w = 1$, $y = 1$, $z = 2$, $u = 2$, $v = 3$.

14. $\frac{x+5}{x-3} > 0$, if $x > 3$ or < -5; $\frac{x+5}{x-3} < 0$, if $-5 < x < 3$.

15. $x < -4$, and $x > 4$. 16. For all real values of x.

Article 59. Page 93

1. 504. 2. 8. 3. 30. 4. $2\frac{1}{4}$. 5. 84. 7. $r^2 - r$.

Article 60. Pages 95–96

6. $\frac{1}{16} + a + 6a^2 + 16a^3 + 16a^4$.

8. $x^5 - 15x^4y^2 + 90x^3y^4 - 270x^2y^6 + 405xy^8 - 243y^{10}$.

9. $a^6 + 6a^5\sqrt{b} + 15a^4b + 20a^3b\sqrt{b} + 15a^2b^2 + 6ab^2\sqrt{b} + b^3$.

10. $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$. 11. $\frac{12}{e} + \frac{40}{e^3} + \frac{12}{e^5}$.

12. $a^4b^4 - 8a^3b^3 + 24a^2b^2 - 32ab + 16$.

13. $x^4 - 8x^{\frac{7}{2}}y^{\frac{1}{2}} + 28x^3y - 56x^{\frac{5}{2}}y^{\frac{3}{2}} + 70x^2y^2 - 56x^{\frac{3}{2}}y^{\frac{5}{2}} + 28xy^3 - 8x^{\frac{1}{2}}y^{\frac{7}{2}} + y^4$.

14. $91\frac{1}{8}$. 15. $a^9 + 6a^{\frac{49}{6}} + 15a^{\frac{22}{3}} + 20a^{\frac{13}{2}} + 15a^{\frac{17}{3}} + 6a^{\frac{29}{6}} + a^4$.

16. $x^{-4} - 4x^{-3}y^{\frac{1}{3}} + 6x^{-2}y^{\frac{2}{3}} - 4x^{-1}y + y^{\frac{4}{3}}$.

17. 32. 18. $\frac{a^6}{b^3} - 6\frac{a^5}{b^{\frac{3}{2}}x^{\frac{1}{2}}} + 15\frac{a^4}{x} - 20\frac{a^3b^{\frac{3}{2}}}{x^{\frac{3}{2}}} + 15\frac{a^2b^3}{x^2} - 6\frac{ab^{\frac{9}{2}}}{x^{\frac{5}{2}}} + \frac{b^6}{x^3}$.

19. $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

20. $x^{\frac{3}{2}} + 6x + 11 + 12x^{\frac{1}{2}} + \frac{12}{x^{\frac{1}{2}}} + \frac{3}{x^{\frac{3}{2}}} + \frac{12}{x} + \frac{6}{x^2} + \frac{1}{x^3}$.

21. $\frac{x^8}{81} + \frac{4}{27}x^6 - \frac{8}{27}x^3 + \frac{2}{3}x^4 + \frac{4}{3}x^2 - \frac{8}{3}x + 1 - \frac{8}{x} + \frac{8}{3x^2} - \frac{8}{x^3} + \frac{16}{x^4} + \frac{24}{x^6} - \frac{32}{3x^7} + \frac{16}{x^{12}} - \frac{32}{x^9}$.

23. $-14{,}080\,a^9b^3$. 24. $2489344\,x^7y^{10}$. 25. $5670\,x^8y^8$.

26. $\frac{18!}{5!\,13!}a^5b^{13}$. 27. $-\frac{13!}{7!\,6!}x^6y^6a^{\frac{21}{2}}$. 28. $126\,x^{\frac{13}{2}}y^7$.

29. $-35\,a^{\frac{6}{5}}$, $+35\,a^{\frac{8}{5}}$.

32. 11,040,808,032. 33. 941,480,149,401. 34. 345,025,251.

35. 0.885842380864. 36. 2.594. 37. 4.177.

Article 64. Pages 98–100

2. $y = 3x$. 3. $s = 16t^2$. 4. $p = \frac{2048}{v}$.

5. $z = 20xy$. 6. $z = \frac{180x}{y}$. 7. $V = kr^3 = \frac{4}{3}\pi r^3$.

8. $V = khr^2$ $= \pi hr^2$ **9.** 402.5 ft. 1610 ft. **10.** 1896+ lb.

11. 25 mi. per hour. **12.** 48 lb. **13.** $b = \frac{kml^3}{wd^3}$.

14. $888\frac{8}{9}$ lb. **15.** 0.12 in. **16.** 7.5 sq ft.

17. 3.4−. **18.** $\frac{25}{128}$ T. **20.** $3\sqrt{5} \pm \sqrt{17}$.

21. 0.087− in. **22.** 1000 yr. **23.** 0.05+ in.

24. 441,720 lb.

Article 68. Pages 102–103

2. $l = -59, s = -610$. **3.** $l = 59, s = 465$. **4.** $l = -71, s = -660$.

5. $l = \frac{1}{12}, s = 3$. **6.** $l = \frac{43}{4}, s = 142$. **7.** $l = \frac{25}{21}, s = \frac{391}{21}$.

8. $n = 7$ or $13, l = 7$ or -5. **9.** $a = 3, s = 465$.

10. $a = -7, s = \frac{-105}{2}$. **11.** $n = 20, d = -4$.

12. $n = 16, s = 142$. **13.** $d = -3, s = -714$.

14. $a = -\frac{3}{7}, n = 23$. **15.** $a = 1, d = 1$.

16. $a = -44, l = 46$. **17.** $l = -5, d = -2$.

19. 5, 8, 11. **20.** $13\frac{1}{3}$. **21.** $\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}$.

Article 72. Page 104

1. $l = 6561, s = 9841$. **2.** $l = -4374, s = -6560$.

3. $l = \frac{1}{1024}, s = \frac{1023}{1024}$. **4.** $l = -768, s = -510$. **5.** $r = 3, l = 162$.

6. 6561. **7.** $\frac{27}{4}$. **8.** 62.

9. $r = 3, s = 120$. **10.** 49. **11.** 16, 128.

12. $\frac{3}{8}$. **13.** $\frac{b^7}{a^5}$.

Article 75. Page 107

1. $\frac{4}{3}$. **2.** $\frac{3}{4}$. **3.** $\frac{4}{5}$. **4.** 16. **5.** $\frac{16}{7}$. **6.** $-\frac{32}{7}$.

7. $\frac{7}{11}$. **8.** $\frac{4}{9}$. **9.** $\frac{5}{6}$. **10.** $\frac{4}{11}$. **11.** $40\frac{10}{11}$. **12.** $\frac{1302}{3300}$.

Article 77. Page 107

1. 4, 6. **2.** $\frac{1}{2}, \frac{2}{7}$. **3.** $\frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}$. **4.** $\frac{2ab}{a+b}$.

Problems. Pages 108–109

1. 30 feet. **2.** $9481\frac{1}{12}$ and 14475 feet.

3. Between 185 and 190 feet per second.

4. 19.683. **5.** 122.853. **6.** 547. **7.** 36.

8. 45. **9.** $885.73. **10.** 2900 feet.

11. The latter proposition is worth $25 per year more to the clerk.

12. 12,092. **13.** $\frac{1}{n}$. **14.** n^2.

15. $x^2 - 14x + 48 = 0$. **17.** $-\frac{7}{3}$. **18.** 6912.

19. 6, 8, 10, 12. **20.** 3, 6, 9. **21.** $7721.73.

Article 80. Page 113

2. $r = \sqrt{13}$, $\theta = \arctan \frac{3}{2}$. **3.** $r = \sqrt{13}$, $\theta = \arctan -\frac{3}{2}$.

4. $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$. **5.** $r = \sqrt{2}$, $\theta = -\frac{\pi}{4}$.

6. $r = \sqrt{41}$, $\theta = \arctan -\frac{4}{5}$. **7.** $r = 1$, $\theta = -\frac{\pi}{2}$.

8. $r = 4$, $\theta = \frac{\pi}{2}$. **9.** $r = 6$, $\theta = -\frac{\pi}{2}$.

10. $r = \frac{3}{2}$, $\theta = \frac{\pi}{2}$. **11.** $r = 4$, $\theta = 0$.

12. $r = 5$, $\theta = \pi$. **13.** $r = \frac{1}{4}\sqrt{13}$, $\theta = \arctan -\frac{2}{3}$.

14. $r = \sqrt{1.7}$, $\theta = \arctan \frac{11}{7}$. **15.** $\frac{3}{2}\sqrt{3} + \frac{3}{2}i$.

16. $1 + i\sqrt{3}$. **17.** $-\sqrt{3} + i$. **18.** $4i$.

19. $-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,i$. **20.** 6. **21.** $-\frac{3}{2}i$.

22. $x = -1$, $y = 1$. **23.** $x = 3$, $y = -1$. **24.** $x = -\frac{13}{25}$, $y = -\frac{7}{25}$.

25. $x = 0$, $y = -1$, or $x = \frac{2}{3}$, $y = -\frac{1}{3}$.

26. $x = 0$, $y = -5$, or $x = 2$, $y = 1$.

27. $x = 9$, $y = 7$, or $x = -9$, $y = -7$.

Article 81. Page 114

1. $3 + 4i$. **2.** $3 - i$. **3.** $1 - 3i$. **4.** 6. **5.** $-10i$.

6. $2 - 5i$. **7.** $1 - i$ **8.** $-1 + 5i$. **9.** $2 - 3i$. **10.** 0.

Article 83. Page 116

2. $4 + 4i\sqrt{3}$, $\theta = \frac{\pi}{3}$, $r = 8$. **3.** $\frac{16}{3}\sqrt{3}\,i$, $\theta = \frac{\pi}{2}$, $r = \frac{16}{3}\sqrt{3}$.

4. $4i$, $\theta = \frac{\pi}{2}$, $r = 4$. **5.** -4, $\theta = \pi$, $r = 4$.

6. $4\sqrt{3} + 4i$, $\theta = \frac{\pi}{6}$, $r = 8$. **7.** -8, $\theta = \pi$, $r = 8$.

8. $-8i$, $\theta = \frac{3\pi}{2}$, $r = 8$. **9.** -10, $\theta = \pi$, $r = 10$.

10. $6i$, $\theta = \frac{\pi}{2}$, $r = 6$. **11.** 4, $\theta = 0$, $r = 4$.

Article 85. Page 118

2. $-432+144i\sqrt{3}$. **3.** $24i\sqrt{3}$. **4.** 1. **5.** 1.

6. $512i$. **7.** $-128-128i$. **8.** $32i$.

10. $-1+i$, $\sqrt{2}(\cos 15^\circ + i\sin 15^\circ)$, $\sqrt{2}(\cos 255^\circ + i\sin 255^\circ)$.

11. $2+2i\sqrt{3}$, $-2-2i\sqrt{3}$.

12. $\sqrt[4]{12}(\cos 15^\circ + i\sin 15^\circ)$, $\sqrt[4]{12}(\cos 195^\circ + i\sin 195^\circ)$.

13. $4\sqrt{3}+4i$, $-4\sqrt{3}-4i$.

14. $4(\cos 20^\circ + i\sin 20^\circ)$, $4(\cos 140^\circ + i\sin 140^\circ)$, $4(\cos 260^\circ + i\sin 260^\circ)$.

15. $2(\cos 10^\circ + i\sin 10^\circ)$, $2(\cos 70^\circ + i\sin 70^\circ)$,
$2(\cos 130^\circ + i\sin 130^\circ)$, $2(\cos 190^\circ + i\sin 190^\circ)$,
$2(\cos 250^\circ + i\sin 250^\circ)$, $2(\cos 310^\circ + i\sin 310^\circ)$.

These results are represented graphically by the six points at equal intervals on a circle of radius 2, beginning with $\theta = 10^\circ$.

16. $\frac{1}{2}\sqrt{3}+\frac{1}{2}i$, $\cos 66^\circ + i\sin 66^\circ$,
$\cos 102^\circ + i\sin 102^\circ$, $\cos 138^\circ + i\sin 138^\circ$,
$\cos 174^\circ + i\sin 174^\circ$, $-\frac{1}{2}\sqrt{3}-\frac{1}{2}i$,
$\cos 246^\circ + i\sin 246^\circ$, $\cos 282^\circ + i\sin 282^\circ$,
$\cos 318^\circ + i\sin 318^\circ$, $\cos 354^\circ + i\sin 354^\circ$.

These results are represented graphically by the ten points at equal intervals on a circle of radius 1, beginning with $\theta = 30^\circ$.

17. $1+i$, $\sqrt{2}(\cos 117^\circ + i\sin 117^\circ)$,
$\sqrt{2}(\cos 189^\circ + i\sin 189^\circ)$, $\sqrt{2}(\cos 261^\circ + i\sin 261^\circ)$,
$\sqrt{2}(\cos 333^\circ + i\sin 333^\circ)$.

18. $\cos 18^\circ + i\sin 18^\circ$, i,
$\cos 162^\circ + i\sin 162^\circ$, $\cos 234^\circ + i\sin 234^\circ$,
$\cos 306^\circ + i\sin 306^\circ$.

19. $\frac{3}{2}\sqrt{3}+\frac{3}{2}i$, $-\frac{3}{2}\sqrt{3}+\frac{3}{2}i$, $-3i$. **20.** ± 2.

21. 1, $-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i$, $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i$. **22.** 2, $-1+i\sqrt{3}$, $-1-i\sqrt{3}$.

23. $3(\cos 22\frac{1}{2}^\circ + i\sin 22\frac{1}{2}^\circ)$,
$3(\cos 112\frac{1}{2}^\circ + i\sin 112\frac{1}{2}^\circ)$, $3(\cos 202\frac{1}{2}^\circ + i\sin 202\frac{1}{2}^\circ)$,
$3(\cos 292\frac{1}{2}^\circ + i\sin 292\frac{1}{2}^\circ)$.

24. 3, $-\frac{3}{2}+\frac{3}{2}i\sqrt{3}$, $-\frac{3}{2}-\frac{3}{2}i\sqrt{3}$.

25. 1, $\cos 72^\circ + i\sin 72^\circ$,
$\cos 144^\circ + i\sin 144^\circ$, $\cos 216^\circ + i\sin 216^\circ$,
$\cos 288^\circ + i\sin 288^\circ$.

26. 2, $2(\cos 72^\circ + i\sin 72^\circ)$,
$2(\cos 144^\circ + i\sin 144^\circ)$, $2(\cos 216^\circ + i\sin 216^\circ)$,
$2(\cos 288^\circ + i\sin 288^\circ)$.

27. See answer to 21. **28.** ± 2, $\pm 2i$.

29. 1, $\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, -1, $-\frac{1}{2}-\frac{1}{2}i\sqrt{3}$, $\frac{1}{2}-\frac{1}{2}i\sqrt{3}$.

30. ± 1, $\pm i$, $\pm\frac{1}{2}\sqrt{2}$, $\pm\frac{1}{2}\sqrt{2}\,i$.

Article 86. Page 120

2. $1+i\sqrt{3}$. **3.** $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$. **4.** $4i$. **5.** $-\frac{1}{4}\sqrt{3}+\frac{1}{4}i$. **6.** $-1+i$.
7. $1-i$. **8.** $-\frac{1}{2}-\frac{1}{2}i$. **9.** i. **10.** $\frac{1}{2}\sqrt{3}-\frac{1}{2}i$.

Article 88. Page 122

3. -1. **4.** $k=-3$.

Article 91. Page 125

2. 0, 2, 4. **3.** 1, 2, 4. **4.** Zero at 1, between 2.5 and 3, and between -2 and -1.5. **5.** Zeros at -2, -1, 2, 3. **6.** Zero between 1.5 and 2, and between -2 and -2.5. **7.** Zero between -1.5 and -1, between -1 and -0.5, between 0 and 0.5, and between 2.5 and 3. **8.** Zero at 2, between 0.5 and 1, and between 2.5 and 3.

Article 97. Page 131

2. (*a*) $x^3-10x^2+31x-30=0$. (*b*) $x^2-2x+5=0$.
(*c*) $x^3-5x^2+5x+3=0$. (*d*) $x^4-5x^2+6=0$.
(*e*) $x^4-2x^3-5x^2+6x=0$. (*f*) $x^4-14x^2+1=0$.

Article 98. Pages 134–135

1. $x^3+10x^2-175x-125=0$. **2.** $x^3-x^2-40=0$.
3. $x^4-10x^2+3x-2=0$. **4.** $x^3+6x^2-80=0$.
5. $x^3-6x^2+3x-270=0$. **6.** $x^4+3x^3-12x^2+20x-8=0$.
7. $x^3+108=0$. **8.** $x^3+6x^2+6x-498=0$.
11. $x^2-x-6=0$. **12.** $x^3+3x^2-4x+1=0$.
13. $x^3-9x^2-90=0$. **14.** $x^5-7x^3+2x-8=0$.
15. $2x^4-3x^2+4x-5=0$.

Article 99. Page 137

2. 4 imaginary roots. **3.** One negative and 4 imaginary roots. **4.** 1 negative and 2 imaginary roots. **5.** 6 imaginary roots. **6.** 1 positive and $n-1$ imaginary roots. **7.** 1 positive, 1 negative, and $n-2$ imaginary roots. **8.** 2 positive and 1 negative roots.

Article 100. Page 138

4. 1. ···, -0. ···, -1. ···. **5.** -1. ···. **6.** 2. ···. **7.** 1. ···, **8.** -1. ···, 11. ···, 0. ···, 0. ···. **9.** 0. ···, 1. ···, 2. ···. **10.** -1. ···, 1. ···, 3. ··· **11.** 2. ···.

Article 102. Pages 140–141

3. $-1, 2, 3, -4$. **4.** $\frac{1}{2}$. **5.** $2, -3, -2$.
6. $-\frac{1}{2}$. **7.** $-\frac{3}{4}$. **8.** $1, \frac{1}{2}, \frac{1}{2}$.
9. 3. **10.** 0. **11.** 1.
12. $-4, 5, 3$. **13.** $2, 6, -7, -1$. **14.** $\frac{1}{4}$.
15. No rational roots. **16.** $\frac{1}{3}$. **17.** $1, 2, -2, 3, 4$.

Article 105. Pages 146–147

1. 3.13^-. **2.** 5.24^-. **3.** 0.64. **4.** 4.64. **5.** 3.98. **6.** $3, 1.41, -1.41$. **7.** 6.17. **8.** $2.36^-, 2.69, -2.05^-$. **9.** -2.21. **10.** 2.24^-. **11.** $3.01, 0.63, -2.02, -0.95$. **12.** $5.83, 0.27, 0.93, -1.03$. **13.** $1.88, -0.35, -1.53$. **14.** $1.18, 2.87$. **15.** -3. **16.** 1.73. **17.** 4.00 per cent. **18.** 0.606. **19.** 0.860. **20.** $0.32, 0.64$. **21.** 6.86. **22.** 11.07. **23** 2.92. **24.** 0.259. **25.** 1.46. **26.** $2\sqrt[3]{\frac{2^9 2}{3}} - 8 = 1.20$, $\frac{5}{2}\sqrt[3]{\frac{2^9 2}{3}} - 10 = 1.50$, $3\sqrt[3]{\frac{2^9 2}{3}} - 12 = 1.80$. **27.** \$1.536.

Article 108. Page 152

3. $-3, \omega, \omega^2$. **4.** $\frac{3}{2}, i, -i$. **5.** $\frac{3}{4}, +\sqrt{2}, -\sqrt{2}$.
6. $\frac{1}{2}, \frac{3}{2}, \frac{3}{2}$. **7.** $2, -1, -1 \pm i\sqrt{2}$. **8.** $-1, \frac{3}{2} \pm \frac{1}{2}i\sqrt{2}$.
9. $-2, -5 \pm 2i\sqrt{3}$. **10.** $\frac{3}{2}, 1, 3 \pm 3\sqrt{2}$.

Article 110. Pages 154–155

1. $2, 2, -2$. **2.** $\frac{3}{2}, \pm i$. **3.** $-1, 2, 5$. **4.** $5, 5, -2$.

5. For $a = 10$, $\{1.218, 0.082\}$; $a = 1$, $\{12.923, 0.077\}$.
$a = 0.1$, $\left\{\begin{matrix}129.923,\\ 0.077,\end{matrix}\right\}$, $a = 0.01$, $\{1299.923, 0.077\}$,
$a = 0.001$, $\{12999.923, 0.077\}$.

6. For $a = 10$, $\{12.179, 0.821\}$, $a = 1$, $\{12.923, 0.077\}$,
$a = 0.1$, $\{12.992, 0.0077\}$, $a = 0.01$, $\{12.999, 0.0008\}$,
$a = 0.001$, $\{13.000^-, 0.000^+\}$.

7. $m = \pm\sqrt{3}$, $b = \mp\frac{2}{3}\sqrt{3}$. **8.** $m = \pm i$, $b = \mp\frac{3i}{2}$. **9.** $m = \pm\frac{3}{4}$, $b = 0$.

10. $m = -3$, $b = -2$, or $m = -\frac{1}{11}$, $b = \frac{1}{11}$.

Article 113. Pages 158–159

2. $2\log 2 - 3\log 3$. **3.** $\frac{1}{2}\log 13 - \frac{1}{5}\log 10 - \frac{1}{3}\log 48$.
4. $\frac{1}{3}\log 25 - \frac{2}{3}\log 11 - \frac{3}{4}\log 23$.
5. $\frac{1}{2}\log 5 + \frac{1}{2}\log 7 - 4\log 2 - 4\log 3 - \frac{3}{4}\log 2 - \frac{1}{2}\log 3$.
6. $-\frac{11}{2}\log 2 - \frac{11}{4}\log 3 - \frac{3}{2}\log 5 - \frac{1}{2}\log 11$.

7. $3 \log 2 + \frac{1}{6} \log 5 - \frac{2}{3} \log 3$. **8.** $\frac{7}{10} \log 2 + \frac{1}{5} \log 3 + \frac{2}{3} \log 7$.

9. $\frac{5}{6} \log 2 + \frac{1}{2} \log 11 + \frac{1}{2} \log 3 + \frac{2}{5} \log 5$.

11. 1.0791. **12.** 1.14771. **13.** 1.6232. **14.** 2.6232.

15. 2.2764. **16.** 2.9542. **17.** 2.5353. **18.** 1.3313.

19. -0.1370. **20.** -2.4013. **21.** -3.0124. **22.** 1.3512.

23. 0.8228. **24.** -0.1505. **25.** 0.6469. **26.** 0.7726.

Article 119. Pages 166–169

1. 0.01359. **2.** 0.2332. **3.** 0.03908. **4.** 24.15.

5. 8.641. **6.** 0.7038. **7.** 1.673. **8.** 0.4352.

9. -0.4704. **10.** 0.2979. **11.** 0.03229. **12.** 0.7295.

13. 0.9630. **14.** 0.3857. **15.** 0.4420. **16.** 64.94.

17. 0.2917. **18.** -0.1606. **19.** 6.636. **20.** 27.89.

21. 25.31. **22.** 3.162. **23.** 0.0696. **24.** 0.9047.

25. -0.8646. **26.** 1.639. **27.** 1.01 sec. **28.** 142.5 tons.

29. Volume = 13,330, Surface = 2719. **30.** $1051 \cdot 10^7$.

31. 11,660. **32.** 834,200. **33.** 1,476,000. **34.** 0.608.

35. 476. **36.** 4.578. **37.** 100 pounds.

38. 0.125 cubic foot. **39.** $p = 4.529$. **40.** 177.5.

41. 0.0068. **42.** 24,470. **43.** \$2014.

44. (*a*) 14,790.
(*b*) 14,860.
(*c*) 14,860.

With a seven-place table the following more accurate results are obtained:
(*a*) 14,802.4.
(*b*) 14,859.4.
(*c*) 14,888.7.

45. \$3767.

46. 41 digits until the year 1935, when it will require 42 digits.

47. 4.251.

48. (1) 100,100; more accurate value 100,081; (2) 85,450; more accurate value, 85,442.

49. 1547 miles.

50. 146,700 sq. kilometers.

Article 120. Page 170

2. 1.3862. **3.** 0.6825. **4.** 2.7302. **5.** 3.322.

6. 1.661. **7.** 2.096. **8.** 1.431. **9.** 3.930.

10. 0.856. **11.** 1.625. **12.** 2.393.

Article 122. Pages 173–175

4. $x = 0.8115$. **5.** $x = 0.1853$. **6.** $x = \pm 2.390$.

7. $x = 1$ or 2. **8.** $x = 3$ or -1.

9. $n = \dfrac{\log rl - \log a}{\log r}$. **10.** $\dfrac{\log [(r-1)s + a] - \log a}{\log r}$.

12. $x = 1.61$, $y = 2.56$. **13.** $x = 6.84$, $y = 10.84$.

14. (1) 11.90, (2) 11.58, (3) 11.55.

With a seven-place table of logarithms, the following more accurate results may be obtained:

(1) 11.89, (2) 11.64, (3) 11.55.

15. $0, \pm 1.32$. **16.** 3.96. **17.** 0.00003776. **18.** 18,360.

19. $k = 0.126$. **20.** 5.5 minutes. **21.** $x = \dfrac{e^2 - 1}{3}$.

22. $x = 25$ and -4. **23.** $s = \dfrac{v_0}{2k}(e^{kt} - e^{-kt})$.

Article 128. Pages 181–182

1. $\dfrac{1}{x-3} - \dfrac{1}{x+2}$. **2.** $\dfrac{2}{x} + \dfrac{5}{3(1-x)} - \dfrac{1}{3(x+2)}$.

3. $\dfrac{3}{4(x-3)} + \dfrac{1}{4(x+1)}$. **4.** $\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} - \dfrac{2}{(x-1)^3} - \dfrac{1}{x+2}$.

5. $\dfrac{3}{x-1} - \dfrac{1}{x-2}$. **6.** $\dfrac{1}{2x} - \dfrac{2}{x-1} + \dfrac{5}{2(x-2)}$.

7. $\dfrac{2}{x+1} + \dfrac{3}{x-1}$. **8.** $\dfrac{1}{x} + \dfrac{2}{(x-1)^2}$.

9. $\dfrac{-2}{x-1} + \dfrac{1}{(x-1)^2} + \dfrac{2x+2}{x^2+1}$. **10.** $x + 1 + \dfrac{1}{x} + \dfrac{1}{x-1} + \dfrac{1}{x+1}$.

11. $\dfrac{5}{2(x-1)} - \dfrac{6}{x-1} + \dfrac{7}{2(x-3)}$. **12.** $\dfrac{2}{x-3} - \dfrac{x+2}{x^2+1} - \dfrac{1}{x+1}$.

13. $\dfrac{2}{x-1} + \dfrac{5}{x-3} - \dfrac{5}{x-2}$. **14.** $\dfrac{1}{x} + \dfrac{2}{x-3} - \dfrac{3}{x+2}$.

15. $-\dfrac{1}{2(x-1)} - \dfrac{3}{x-2} + \dfrac{9}{2(x-3)}$. **16.** $\dfrac{1}{x} - \dfrac{x}{x^2+4}$.

17. $\dfrac{4}{3(x-1)} - \dfrac{4x+8}{3(x^2+x+1)}$. **18.** $-\dfrac{2}{x} - \dfrac{1}{x^2} - \dfrac{1}{x^3} + \dfrac{2}{x-1}$.

19. $\dfrac{1}{1-10x} + \dfrac{1}{3(1+3x)} - \dfrac{1}{3(1+3x)^2}$.

20. $\dfrac{11}{8(x-3)} - \dfrac{7}{2(x-1)^3} - \dfrac{11}{4(x-1)^2} - \dfrac{11}{8(x-1)}$.

21. $\frac{3}{2(1-3x)^3}+\frac{21}{8(1-3x)^2}+\frac{21}{32(1-3x)}+\frac{7}{32(1+x)}$.

22. $\frac{1}{x+2}+\frac{2-x}{x^2-2x+5}$.

23. $\frac{1}{x-1}+\frac{1}{(x^2-x+1)^2}$.

24. $\frac{2}{x+1}+\frac{2}{x-1}+\frac{2x+2}{x^2+1}$.

25. $\frac{1}{x+1}-\frac{2}{x-1}+\frac{3x-4}{x^2+x+1}$.

26. $\frac{1}{2(2x-3)}-\frac{27}{2(2x-3)^3}-\frac{27}{(2x-3)^4}$.

27. $\frac{5}{x}-\frac{1}{x^2}-\frac{5}{x+1}-\frac{4}{(x+1)^2}$.

28. $\frac{2}{x-2}-\frac{3}{x+2}-\frac{x-10}{x^2+4}$.

29. $x+2+\frac{3}{x+1}+\frac{4}{x-5}$.

30. $\frac{1}{x-3}-\frac{x+2}{x^2-x+1}-\frac{2x+3}{(x^2-x+1)^2}$.

31. $\frac{1}{x^2}-\frac{2}{x}+\frac{8}{x-1}-\frac{5}{(x-1)^2}$.

32. $3+\frac{1}{x+2}-\frac{5}{(x+2)^2}-\frac{18}{(x+2)^3}$.

33. $\frac{1}{x+3}+\frac{2}{x-3}-\frac{3x+4}{x^2+3}$.

34. $\frac{4}{(x-1)^2}-\frac{1}{x-1}+\frac{x-3}{x^2-x+1}$.

35. $\frac{1}{x^2+x+2}-\frac{1}{x^2+x+1}+\frac{x}{(x^2+x+2)^2}$.

36. $\frac{x+2}{x^2+1}-\frac{x}{(x^2+1)^2}$.

37. $\frac{2}{1-x}+\frac{1}{(1-x)^3}+\frac{2x+4}{1+x+x^2}$.

38. $\frac{1}{3x+2}+\frac{x^3+2x^2+2}{(x^2+1)^2}$.

Article 132. Pages 185–186

1. 720. **2.** 5040. **3.** 60. **4.** 1,860,480.
5. 40,320. **6.** 8. **7.** 3360; 34,650. **8.** 5040.
9. 336. **10.** 360. **11.** 30. **12.** 180.

Article 136. Pages 187–189

1. 210. **2.** 40. **3.** 31.
4. 502,501. **5.** 70. **6.** 1140; 220.
7. 10. **8.** $n=17$, $r=2$. **9.** 31.
10. 63. **11.** 25. **12.** 462.
13. 371. **15.** 4950. **16.** 327,600.
17. 45. **18.** $\frac{52\,!\,4\,!}{(13\,!)^4}$. **19.** 20.
20. 30,240. **21.** 3360. **22.** 4095.
23. 1820; 455. **24.** 10,080. **25.** 3,303,300.
26. $\frac{100\,!}{60\,!\,20\,!\,20\,!}$.

Article 139. Pages 191–192

1. $\frac{10}{11}$. 3. $\frac{10}{21}$. 4. $\frac{1}{78}$. 5. \$5. 6. 0.69804. 7. (1) $\frac{4}{33}$; (2) $\frac{5}{33}$; (3) $\frac{1}{99}$. 8. \$3.

Article 140. Page 193

1. $\frac{7}{12}$. 2. $\frac{11}{36}$. 3. $\frac{1}{20}$. 4. $\frac{2}{11}$. 5. $\frac{7}{9}$. 6. $\frac{10}{21}$. 7. $\frac{5}{12}$. 8. $\frac{8}{27}$.
9. (1) 0.03; (2) 0.68; (3) 0.12; (4) 0.17.

Article 141. Pages 194–195

1. (1) $\frac{5}{16}$; (2) $\frac{21}{32}$.
2. (1) 0.2886; (2) 0.9446.
3. $\frac{161}{163}\frac{178}{184}$.
4. $\frac{53}{64}$.
5. \$10.
6. (1) 0.108; (2) 0.994.
7. The latter.
8. (1) 0.087; (2) 0.503.
10. $\frac{3}{676}$.
11. $\frac{31}{63}$.
12. (1) $\frac{1}{8}$; (2) $\frac{255}{2116}$.
13. A's expectation is \$36; B's is \$30.
14. (1) $\frac{64,763}{75,994,575} = 0.000852$; (2) $\frac{89,827}{75,994,575} = 0.00117$; (3) 0.00000100; (4) 76.

Article 144. Page 202

2. -25. 3. 5. 4. 2. 5. -99.

Article 149. Pages 211–212

1. 2, 1.
2. 1, 0, -1.
3. $\frac{1}{2}$, ∞, 1.
4. 2, 4, -1, -3.
5. $k = 4$.
6. $x : y : z = -1 : 1 : 2$.
7. $5x^6 - 2x^4 - 9x^2 + 6x - 1 = 0$.
8. $a^3 + b^3 + c^3 - 3abc = 0$.
9. Yes.
10. 2, or $-1\frac{5}{3}\frac{4}{3}$.

Article 156. Page 220

1. 3.
2. $-\frac{1}{4}$.
3. $2\sqrt{2}$.
4. $\frac{5}{2}$.
5. $-\frac{3a}{2}$.
6. 4.
7. $\frac{1}{2}$.
8. 1.
9. Increases without limit when $x \to -1$.
10. 1.
11. 0.
12. $\frac{1}{2}$.

Article 163. Page 231

1. Convergent.
2. Convergent.
3. Convergent.
4. Divergent.
5. Divergent.
6. Convergent.
7. Convergent.
8. Divergent.
9. Convergent.

Article 167. Pages 235–236

1. Convergent. 2. Convergent. 3. Convergent.
4. Convergent. 5. Convergent. 6. Convergent.
7. Divergent. 8. Divergent. 9. Convergent.
10. Divergent. 11. 0.8965. 12. 0.4794.
13. 0.7834. 14. 0.2877. 15. Convergent.
16. Convergent. 17. Divergent. 18. Convergent.
19. Divergent for all values of x.
20. Convergent for $|x| < 1$, also for $x = -1$.
21. Convergent for $x > 0$, and $x < -1$. 22. Convergent for $|x| < 1$.
23. Convergent for $x > -1$ and $x < -3$.

Article 168. Pages 237–238

2. Convergent for $|x| < \frac{1}{2}$. 3. Convergent for $|x| < 3$.
4. Divergent for all values of x, for which the series is defined.
5. Convergent for $-\infty < x < \infty$. 6. Convergent for $-\infty < x < \infty$.
7. Convergent for $|x| < 1$. 8. Convergent for $|x| < 1$.
9. Convergent for $|x| < \frac{1}{4}$. 10. Convergent for $|x| < \frac{2}{3}$.
11. Convergent for $|x| < 1$. 12. Convergent for $-\infty < x < \infty$.
13. Convergent for $-\infty < x < \infty$.
14. Convergent for $-1 < x < 1$ and for $x = -1$.
15. Convergent for $-1 \leqq x \leqq 1$. 16. 0.0872–.

Article 169. Pages 239–240

1. $1 - 6x + 27x^2 - 108x^3 + 405x^4 -$. $|x| < \frac{1}{3}$.
2. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \cdots$. $|x| < \frac{1}{2}$.
3. $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$. $|x| < 1$.
4. $\frac{\sqrt{2}}{2}(1 + \frac{3}{4}x + \frac{27}{32}x^2 + \frac{135}{128}x^3 + \frac{2835}{2048}x^4 + \cdots)$. $|x| < \frac{2}{3}$.

12. 2.236. 13. 3.873. 14. 1.010. 15. 9.993. 16. 2.005.
17. 5.066. 18. 9.008. 19. 2.002. 20. 0.8016.

Article 171. Page 242

2. 0.2877. 3. $\log_e 16 = 2.772$. $\log_e 17 = 2.833$.

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Binomial Theorem!

Coef. of 1st term = no of power to be raised to.

" " 2nd " =

r^{th} term $= \frac{n \cdot n-1 \cdots n-r+2}{r-1!} a^{n-r+1} \times r-1$

$a^0 = 1$

$a^n \cdot a^0 = a^n$ addition

$a^{-n} = \frac{1}{a^n}$

proving that $a^{-n} = \frac{1}{a^n}$

$a^n \cdot a^{-n} = a^0 = \frac{1}{a^n} = \frac{1}{a^n}$

$\frac{a^n}{a^n}$

Cancelling a^n:

$f(x) = x^3 + 2x^2 - 6$

$f(n) = n^3 + 2n^2 - 6$

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} - \dots + a_{n-1} x + a_n$

$f(n) = a_0 n^n + a_1 n^{n-1} + a_2 n^{n-2} - \dots + a_{n-1} n + a_n$

$f(x) - f(n) = a_0(x^n - n^n) + a_1(x^{n-1} - n^{n-1}) \dots a_{n-1}(x-n)$

$f(x) - f(n) = (x-n)[a_0(\ ) + a_1(\ ) + a_{n-1}]$

$f(x) - f(n) = (x-n)\,Q(x)$

$\frac{f(x)}{x-n} = \frac{(x-n)\,Q(x)}{x-n} + \frac{f(n)}{x-n}$

$\frac{f(x)}{(x-n)} = Q(x) + \frac{f(n)}{x-n}$ Schema

application

$\frac{f(x)}{x-5} = Q(x) + \frac{14}{x}$

$f(x) = x^3 - 6x^2 + 2x + 1$

$x - n = x - 5$

$f(5) = 5^3 - 6 \cdot 5^2 + 2 \cdot 5 + 1$

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